Malcolm Westergaard
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INTRODUCTORY COURSE
IN
DIFFERENTIAL EQUATIONS

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DIFFERENTIAL EQUATIONS

FOR STUDENTS IN CLASSICAL AND ENGINEERING COLLEGES

BY

DANIEL A. MURRAY, PH.D.
PROFESSOR OF APPLIED MATHEMATICS IN MCGILL UNIVERSITY

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PREFACE.

THE aim of this work is to give a brief exposition of some of the devices employed in solving differential equations. The book presupposes only a knowledge of the fundamental formulæ of integration, and may be described as a chapter supplementary to the elementary works on the integral calculus.

The needs of two classes of students, with whom the author has been brought into contact in the course of his experience as a teacher, have determined the character of the work. For the sake of students of physics and engineering who wish to use the subject as a tool, and have little time to devote to general theory, the theoretical explanations have been made as brief as is consistent with clearness and sound reasoning, and examples have been worked in full detail in almost every case. Practical applications have also been constantly kept in mind, and two special chapters dealing with geometrical and physical problems have been introduced.

The other class for which the book is intended is that of students in the general courses in Arts and Science, who have more time to gratify any interest they may feel in this subject, and some of whom may be intending to proceed to the study of the higher mathematics. For these students, notes have been inserted in the latter part of the book. Some of the notes contain the demonstrations of theorems which are referred to, or partially proved, in the first part of the work. If these discussions were given in full in the latter place, they would probably tend to discourage a beginner. Accordingly, it has been thought better to delay the rigorous proof of several theorems until the student has acquired some degree of familiarity with the working of examples.

Throughout the book are many historical and biographical notes, which it is hoped will prove interesting. In order that beginners may have a larger and better conception of the subject, it seemed right to point out to them some of the most important lines of development of the study of differential equations, and notes have been given which have this object in view. For this purpose, also, a few articles have been placed in the body of the text. These articles refer to Riccati's, Bessel's, Legendre's, Laplace's, and Poisson's equations, and the equation of the hypergeometric series, which are forms that properly lie beyond the scope of an introductory work.

In many cases in which points are discussed in the brief manner necessary in a work of this kind, references are given where fuller explanations and further developments may be found. These references are made, whenever possible, to sources easily accessible to an ordinary student, and especially to the standard treatises, in English, of Boole, Forsyth, and Johnson.

For students who can afford but a minimum of time for this study, the essential articles of a short course are indicated after the table of contents.

Of the examples not a few are original, and many are taken from examination papers of leading universities. There is also a large number of examples, which, either by reason of their frequent use in mechanical problems or their excellence as examples *per se*, are common to all elementary text-books on differential equations.

There remains the pleasant duty of making confession of my indebtedness.

In preparing this book, I have consulted many works and memoirs; and, in particular, have derived especial help for the principal part of the work from the treatises of Boole, Forsyth, and Johnson, and from the chapters on Differential Equations in the works of De Morgan, Moigno, Hoüel, Laurent, Boussinesq, and Mansion. I have in addition to acknowledge suggestions received from Byerly's "Key to the Solution of Differential Equations" published in his *Integral Calculus*, Osborne's *Examples and Rules*, and from the treatises of Williamson, Edwards, and Stegemann on the Calculus. Use has also been made of notes of a course of lectures delivered by Professor David Hilbert at Göttingen. Suggestions and material for many of the historical and other notes have also been received from the works of Craig, Jordan, Picard, Goursat, Koenigsberger, and Schlesinger on Differential Equations; from Byerly's *Fourier's Series and Spherical Harmonics*, Cajori's *History of Mathematics*, and from the chapters on Hyperbolic Functions, Harmonic Functions, and the History of Modern Mathematics in Merriman and Woodward's *Higher Mathematics.* The mechanical and physical examples have been obtained from Tait and Steele's *Dynamics of a Particle*, Ziwet's *Mechanics*, Thomson and Tait's *Natural Philosophy*, Emtage's *Mathematical Theory of Electricity and Magnetism*, Bedell and Crehore's *Alternating Currents*, and Bedell's *Principles of the Transformer*. These and many other acknowledgments will be found in various parts of the book.

To the friends who have encouraged and aided me in this undertaking, I take this opportunity of expressing my gratitude. And first and especially to Professor James McMahon of Cornell University, whose opinions, advice, and criticisms, kindly and freely given, have been of the greatest service to me. I have also to thank Professors E. Merritt and F. Bedell of the department of physics, and Professor Tanner, Mr. Saurel, and Mr. Allen of the department of mathematics at Cornell for valuable aid and suggestions. Professor McMahon and Mr. Allen have also assisted me in revising the proof-sheets while the work was going through the press. To Miss H. S. Poole and Mr. M. Macneill, graduate students at Cornell, I am indebted for the verification of many of the examples.

D. A. MURRAY.

Cornell University,
April, 1897.

PREFACE TO THE SECOND EDITION.

I take this opportunity of expressing my thanks to my fellow-teachers of mathematics for the kind reception which they have given to this book. My gratitude is especially due to those who have pointed out errors, made criticisms, or offered suggestions for improving the work. Several of these suggestions have been adopted in preparing this edition. It is hoped that the answers to the examples are now free from mistakes.

D. A. MURRAY.

Cornell University,
June, 1898.

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SHORT COURSE.

(The Roman numerals refer to chapters, the Arabic to articles.)

I.; II. 7–16, 20, 21; III.; IV. 30–34; V.; VI. 49–53, 56–62; VII. 65, 66, 71; VIII. 72–81; IX. 84, 85, 87, 90–93; X.; XI. 97–99, 101–103, 106; XII. 107–116, 119–122, 124, 125, 127, 128, 131, 133.

DIFFERENTIAL EQUATIONS.

CHAPTER I.

DEFINITIONS. FORMATION OF A DIFFERENTIAL EQUATION.

1. Ordinary and partial differential equations. Order and degree. *A differential equation* is an equation that involves differentials or differential coefficients.

Ordinary differential equations are those in which all the differential coefficients have reference to a single independent variable. Thus,

$$dy = \cos x\, dx, \tag{1}$$

$$\frac{d^2y}{dx^2} = 0, \tag{2}$$

$$(y+c)^2\frac{dx}{dz} + z\frac{dy}{dz} - (y+a) = 0, \tag{3}$$

$$y = x\frac{dy}{dx} + r\sqrt{1+\left(\frac{dy}{dx}\right)^2}, \tag{4}$$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = r, \tag{5}$$

$$y = x\frac{dy}{dx} + \frac{a}{\frac{dy}{dx}}, \tag{6}$$

are ordinary differential equations.

Partial differential equations are those in which there are two or more independent variables and partial differential coefficients with reference to any of them; as,

$$y^2\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=nxz.$$

The *order* of a differential equation is the order of the highest derivative appearing in it.

The *degree* of an equation is the degree of that highest derivative, when the differential coefficients are free from radicals and fractions. Of the examples above, (1) is of the first order and first degree, (2) is of the second order and first degree, (4) is of the first order and second degree, (5) is of the second order and second degree, (6) is of the first order and second degree. In the integral calculus a very simple class of differential equations of which (1) is an example have been treated.

Equations having one dependent variable y and one independent variable x will first be considered. The typical form of such equations is

$$f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^ny}{dx^n}\right)=0.$$

2. Solutions and constants of integration. Whether a differential equation has a solution, what are the conditions under which it will have a solution of a particular character, and other questions arising in the general theory of the subject are hardly matters for an introductory course.* The student will remember that he solved algebraic equations, before he could prove that such equations must have roots, or before he had more than a very limited knowledge of their general properties. This book will be concerned merely with an exposition of the methods of solving some particular classes of differential equations; and their solutions will be expressed by the ordinary algebraic, trigonometric, and exponential functions.

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* For a proof that a differential equation has an integral, and for references relating to this fundamental theorem, see Note B, p. 190.

A *solution* or *integral* of a differential equation is a relation between the variables, by means of which and the derivatives obtained therefrom, the equation is satisfied.

Thus $y = \sin x$ is a solution of (1);

$$x^2 + y^2 = r^2 \text{ and } y = mx + r\sqrt{1 + m^2}$$

are solutions of (4) Art. 1.* In two of these solutions, y is expressed explicitly in terms of x, but in the solutions of differential equations in general, the relation between x and y is oftentimes not so simply expressed. This will be seen by glancing at the solutions of the examples on Chapter II.

A solution of (1) Art. 1 is $y = \sin x$; another solution is

$$y = \sin x + c, \tag{1}$$

c being any constant. By changing the value of c, different solutions are obtained, and in particular, by giving c the value zero, the solution $y = \sin x$ is obtained.

A solution of
$$\frac{d^2y}{dx^2} + y = 0 \tag{2}$$

is $y = \sin x$, and another solution is $y = \cos x$. A solution more general than either of the former is $y = A \sin x$; and it includes one of them, as is seen by giving A the particular value unity. Similarly $y = B \cos x$ includes one of the two first given solutions of (2). The relation

$$y = A \cos x + B \sin x \tag{3}$$

is a yet more general solution, from which all the preceding solutions of (2) are obtainable by giving particular values to A and B.

The arbitrary constants A, B, c, appearing in these solutions are called *arbitrary constants of integration*.

Solution (1) has one arbitrary constant, and solution (3) has two; the question arises: *How many* arbitrary constants must the most general solution of a differential equation contain?

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* See page 12.

The answer can in part be inferred from the consideration of the formation of a differential equation.

3.* **The derivation of a differential equation.** In the process of deriving (2) from (3) Art. 2, A and B have been made to disappear. To eliminate two constants, A and B, three equations are required. Of these three equations, one is given, namely, (3), and the two others needed are obtained by successive differentiation of (3). Thus,

$$y = A\sin x + B\cos x,$$

$$\frac{dy}{dx} = A\cos x - B\sin x,$$

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x;$$

whence,

$$\frac{d^2y}{dx^2} + y = 0.$$

Now consider the general process. The equation

$$f(x, y, c_1, c_2, \cdots, c_n) = 0 \tag{1}$$

contains, besides x and y, n arbitrary constants $c_1, c_2, \cdots, c_n$. Differentiation n times in succession with respect to x gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0.$$

$$\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\,\partial y}\frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial f}{\partial y}\frac{d^2y}{dx^2} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{\partial^n f}{\partial x^n} + \cdot \quad \cdot \quad \cdot \quad \cdot \quad + \frac{\partial f}{\partial y}\frac{d^ny}{dx^n} = 0.$$

Between the original equation and the n equations thus obtained by differentiation, making $n+1$ equations in all, the

---

* See B. Williamson, *Differential Calculus*, Art. 311; J. Edwards, *Differential Calculus*, Arts. 506, 507.

n constants $c_1, c_2, \cdots, c_n$ can be eliminated, and thus will be formed the equation

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0. \tag{2}$$

Therefore, when there is a relation between x and y involving n arbitrary constants, the corresponding differential relation which does not contain the constants is obtained by eliminating these n constants from the $n+1$ equations, made up of the given relation and n new equations arising from n successive differentiations. There being n differentiations, the resulting equation must contain a derivative of the nth order, and therefore a relation between x and y, involving n arbitrary constants, will give rise to a differential equation of the nth order free from those constants. The equation obtained is independent of the order in which, and of the manner in which, the eliminations are effected.*

On the other hand, it is evident that a differential equation of the nth order cannot have more than n arbitrary constants in its solution; for, if it had, say $n+1$, on eliminating them there would appear, not an equation of the nth order, but one of the $(n+1)$th order.†

Ex. 1. From $x^2 + y^2 + 2ax + 2by + c = 0,$

derive a differential equation not containing a, b, or c.

Differentiation three times in succession gives

$$x + y\frac{dy}{dx} + a + b\frac{dy}{dx} = 0,$$

$$1 + \left(\frac{dy}{dx}\right)^2 + y\left(\frac{d^2y}{dx^2}\right) + b\left(\frac{d^2y}{dx^2}\right) = 0,$$

$$3\frac{dy}{dx}\frac{d^2y}{dx^2} + y\frac{d^3y}{dx^3} + b\frac{d^3y}{dx^3} = 0.$$

---

* See Joseph Edwards, *Differential Calculus*, Art. 507, after reading Arts. 5, 6, following.

† For a proof that the general solution of an equation of the nth order contains exactly n arbitrary constants, see Note C, p. 194.

The elimination of b from the last two equations gives the differential equation required,

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0.$$

Ex. **2.** Form the differential equation corresponding to

$$y^2-2\,ay+x^2=a^2,$$

by eliminating a.

Ex. **3.** Eliminate α and β from $(x-\alpha)^2+(y-\beta)^2=r^2$.

Ex. **4.** Eliminate m and a from $y^2=m(a^2-x^2)$.

4. Solutions, general, particular, singular. The solution which contains a number of arbitrary constants equal to the order of the equation, is called *the general solution* or *the complete integral.* Solutions obtained therefrom, by giving particular values to the constants, are called *particular solutions.* Looking on the differential equation as derived from the general solution, the latter is called *the complete primitive* of the former.

It may be noted that from the relation (1) Art. 3 several differential equations can be derived, which are different when the constants chosen to be eliminated are different. Thus, the elimination of all the constants gives but one differential equation, namely (2), for the order of elimination does not affect the equation formed. The elimination of all but c_1 gives an equation of the $(n-1)$th order; elimination of all but c_2 gives another equation of the $(n-1)$th order; and similarly for $c_3, \cdots, c_n$. So from (1), n equations of the $(n-1)$th order can be derived. Therefore (1) is the complete primitive of one equation of the nth order, and the complete primitive of n different equations of the $(n-1)$th order. The student may determine how many equations of the first, second, $\cdots$, $(n-2)$th order can be derived from (1).

The general solution may not include all possible solutions. For instance, (4) Art. 1 has for solutions, $x^2+y^2=r^2$, and $y=mx+r\sqrt{1+m^2}$. The latter is the general solution, containing the arbitrary constant m, but the former is not derivable from it by giving particular values to m. It is called a *singular solution*. Singular solutions are discussed in Chapter IV.

The n arbitrary constants in the general solution must be independent and not equivalent to less than n constants. The solution $y = ce^{x+\alpha}$ appears to contain two arbitrary constants c and α, but they are really equivalent to only one, for

$$y = ce^{x+\alpha} = ce^{\alpha}e^{x} = Ae^{x},$$

and by giving A all possible values, all the particular solutions, that can be obtained by giving c and α all possible values, will also be obtained.*

The general solution can have various forms, but there will be a relation between the arbitrary constants of one form and those of another. For example, it has been seen that the general solution of $\frac{d^2y}{dx^2} + y = 0$ is

$$y = A\sin x + B\cos x.$$

But $y = c\sin(x + \alpha)$ is also a solution, as may be seen by substitution in the given equation; and it is a general solution, since it contains two independent constants c and α. The latter form expanded is

$$y = c\cos\alpha\sin x + c\sin\alpha\cos x.$$

On comparing this form with the first form of solution given, it is evident that the relations between the constants A, B, of the first form and c, α, of the second, are

$$A = c\cos\alpha, \text{ and } B = c\sin\alpha,$$

that is,

$$c = \sqrt{A^2 + B^2}, \text{ and } \alpha = \tan^{-1}\frac{B}{A}.$$

If the solution has to satisfy other conditions besides that made by the given differential equation, some or all of the constants will have determinate values, according to the number of conditions imposed.

---

* See Note D for a criterion of the independence of the constants.

5. Geometrical meaning of a differential equation of the first order and degree.

Take
$$f\left(x,\ y,\ \frac{dy}{dx}\right)=0, \tag{1}$$

an equation of the first degree in $\frac{dy}{dx}$. It will be remembered, that when the equation of a curve is given in rectangular co-ordinates, the tangent of its direction at any point is $\frac{dy}{dx}$. For any particular point (x_1, y_1), there will be a corresponding particular value of $\frac{dy}{dx}$, say m_1, determined by equation (1). A point that moves, subject to the restriction imposed by this equation, on passing through (x_1, y_1) must go in the direction m_1. Suppose it moves from (x_1, y_1) in the direction m_1 for an infinitesimal distance, to a point (x_2, y_2); then, that it moves from (x_2, y_2) in the direction m_2, the particular direction associated with (x_2, y_2) by the equation, for an infinitesimal distance to a point (x_3, y_3); thence, under the same conditions to (x_4, y_4), and so on through successive points. In proceeding thus, the point will describe a curve, the co-ordinates of every point of which, and the direction of the tangent thereat, will satisfy the differential equation. If the moving point starts at any other point, not on the curve already described, and proceeds as before, it will describe another curve, the co-ordinates of whose points and the direction of the tangents thereat satisfy the equation. Through every point on the plane, there will pass a particular curve, for every point of which, x, y, $\frac{dy}{dx}$, will satisfy the equation. The equation of each curve is thus a particular solution of the differential equation; the equation of the system of such curves is the general solution; and all the curves represented by the general solution, taken together, make *the locus of the differential equation.* There being one arbitrary constant in the general solution of an equation of the first order, the locus of the latter is made up of a single infinity of curves.

Ex. 1. The equation $\frac{dy}{dx} = -\frac{x}{y}$

indicates that a point moving so as to satisfy this equation, moves perpendicularly to the line joining it to the origin; that is, it describes a circle about the origin as centre.

Putting the equation in the form

$$x\,dx + y\,dy = 0,$$

it is seen that the general solution is

$$x^2 + y^2 = c.$$

The circle passing through a particular point, as (3, 4), is

$$x^2 + y^2 = 25,$$

which is a particular solution. The general solution thus represents the system of circles having the origin for centre, and the equation of each one of these circles is a particular solution. That is, the locus of the differential equation is made up of all the circles, infinite in number, that have the origin for centre.

Ex. 2. $x\,dy + y\,dx = 0$

has for its solution, $xy = c$,

the equation of the system of hyperbolas, infinite in number, that have the x and y axes for asymptotes.

Ex. 3. $\frac{dy}{dx} = m$,

having for its solution, $y = mx + c$,

has for its locus all straight lines, infinite in number, of slope m.

6. Geometrical meaning of a differential equation of a degree or an order higher than the first.

If $f\left(x, y, \frac{dy}{dx}\right) = 0$

is of the second degree in $\frac{dy}{dx}$, there will be two values of $\frac{dy}{dx}$ belonging to each particular point (x_1, y_1). Therefore the moving point can pass through each point of the plane in either of two directions; and hence, two curves of the system which is the locus of the general solution pass through each point. The general solution,

$$\phi(x, y, c) = 0,$$

must therefore have two different values of c for each point; and hence, c must appear in that solution in the second degree.

In general, it may be said: A differential equation,

$$f\left(x, y, \frac{dy}{dx}\right) = 0,$$

which is of the nth degree in $\frac{dy}{dx}$, and which has

$$\phi(x, y, c) = 0$$

for its general solution, has for its locus a single infinity of curves, there being but one arbitrary constant in ϕ; n of these curves pass through each point of the plane, since $\frac{dy}{dx}$ has n values at any point; and hence the constant c must appear in the nth degree in the general solution.

The general solution of a differential equation of the second order,

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0,$$

contains two arbitrary constants, and will therefore have for its locus a double infinity of curves; that is, a set of curves ∞^2 in number.

Ex. 1. $$\frac{d^2y}{dx^2} = 0$$

has for its solution, $$y = mx + c,$$

m and c being arbitrary.

A line through any point $(0, c)$, drawn in any direction m, is the locus of a particular integral of the equation. On taking a particular value of c, say c_1, there will be an infinity of lines corresponding to the infinity of values that m can have, and all these lines are loci of integrals. Since to each of the infinity of values that c can have there corresponds an infinity of lines, the complete integral will represent a doubly infinite system of straight lines; in other words, the locus of that differential equation consists of a doubly infinite system of lines.

This can be deduced from other considerations. The condition $\frac{d^2y}{dx^2}=0$ requires, and requires only, that the curve described by the moving point shall have zero curvature, that is, it can be any straight line; and there can be ∞^2 straight lines drawn on a plane.

Ex. 2. All circles of radius r, ∞^2 in number, are represented by the equation

$$(x-a)^2+(y-b)^2=r^2,$$

where a and b, the co-ordinates of the centre, are arbitrary. On eliminating a and b, there appears

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}=r\frac{d^2y}{dx^2}.$$

Thus, the locus of the latter equation of the second order consists of the doubly infinite system of circles of radius r.

Ex. 3. The locus of the differential equation of the third order, derived in Example 1, Art. 3, includes all circles, ∞^3 in number; for it is derived from a complete primitive which has a, b, c arbitrary and thus represents circles whose centres and radii are arbitrary.

It will have been observed from the above examples on lines and circles, that as the order of the differential equation rises, its locus assumes a more general character.

EXAMPLES ON CHAPTER I.

1. Eliminate the constant a from $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$.

2. Form the differential equation of which $y=ce^{\sin^{-1}x}$ is the complete integral.

3. Find the differential equation corresponding to

$$y=ae^{2x}+be^{-3x}+ce^{x},$$

where a, b, c are arbitrary constants.

4. Form the differential equation of which $c(y+c)^2=x^3$ is the complete integral.

5. Eliminate c from $y=cx+c-c^3$.

6. Eliminate c from $ay^2 = (x - c)^3$.

7. Form the differential equation of which $e^{2y} + 2\,cxe^{y} + c^2 = 0$ is the complete integral.

8. Eliminate a and b from $xy = ae^{x} + be^{-x}$.

9. Form the differential equation which has $y = a\cos(mx + b)$ for its complete integral, a and b being the arbitrary constants.

10. Form the differential equation that represents all parabolas each of which has a latus rectum $4\,a$, and whose axes are parallel to the x axis.

11. Find the differential equation of all circles which pass through the origin and whose centres are on the x axis.

12. Form the differential equation of all parabolas whose axes are parallel to the axis of y.

13. Form the differential equation of all conics whose axes coincide with the axes of co-ordinates.

14. Eliminate the constants from $y = ax + bx^2$.

Note. [The following is intended to follow line 6, page 3.]

Ex. **1.** Show that $x^2 + y^2 = r^2$ is a solution of equation (4) Art. 1.

Differentiation gives $x + y\dfrac{dy}{dx} = 0$, whence $\dfrac{dy}{dx} = -\dfrac{x}{y}$.

Substitution of this value of $\dfrac{dy}{dx}$ in 4 gives $y = -\dfrac{x^2}{y} + r\sqrt{1 + \dfrac{x^2}{y^2}}$,

which reduces to $x^2 + y^2 = r^2$.

Ex. **2.** Show that $y = mx + r\sqrt{1 + m^2}$ is a solution of (4) Art. 1.

Differentiation gives $\dfrac{dy}{dx} = m.$

Substitution of this value of $\dfrac{dy}{dx}$ in (4) gives $y = mx + r\sqrt{1 + m^2}$.

Ex. **3.** Show that $x^2 + 4\,y = 0$ is a solution of $\left(\dfrac{dy}{dx}\right)^2 + x\dfrac{dy}{dx} - y = 0$.

Ex. **4.** Show that $y = ax^2 + bx$ is a solution of $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\dfrac{dy}{dx} + \dfrac{2\,y}{x^2} = 0$.

Ex. **5.** Show that $v = \dfrac{A}{r} + B$ is a solution of $\dfrac{d^2v}{dr^2} + \dfrac{2}{r}\dfrac{dv}{dr} = 0$.

Ex. **6.** Show that $y = ae^{kx} + be^{-kx}$ is a solution of $\dfrac{d^2y}{dx^2} - k^2y = 0$.

CHAPTER II.

EQUATIONS OF THE FIRST ORDER AND OF THE FIRST DEGREE.

7. In Chapter I. it has been shown how to deduce from a given relation between x, y, and constants, a relation between x, y, and the derivatives of y with respect to x. There has now to be considered the inverse problem: viz., from a given relation between x, y, and the derivatives of y, to find a relation between the variables themselves. As, for instance, the problem of finding the roots of an algebraic equation is more difficult than that of forming the equation when the roots are given; or as, again, integration is a more difficult process than differentiation; so here, as in other inverse processes, the process of solving a differential equation is much more complicated and laborious than the direct operation of forming the equation when the general solution is given. An equation is said to be solved, when its solution has been reduced to expressions of the forms $\int f(x)dx$, $\int \phi(y)dy$, even if it be impossible to evaluate these integrals in terms of known functions.

The equation $f\left(x, y, \frac{dy}{dx}, \cdots, \frac{d^n y}{dx^n}\right) = 0$ cannot be solved in every case. In fact, even $P\frac{dy}{dx} + Q = 0$, where P and Q are functions of x and y, cannot be solved completely. It will be remembered how few in number are the solvable cases in algebraic equations; and it is the same with differential equations. The remainder of this book will be taken up with a consideration of *a few special forms of equations and the methods devised for their solution.**

This chapter will be devoted to certain kinds of equations of the first order and degree, viz.:

1. Those that are either of the form $f_1(x)\,dx + f_2(y)\,dy = 0$, or are easily reducible to this form;
2. Those that are reducible to this form by the use of special devices:—
   (*a*) Equations homogeneous in x and y.
   (*b*) Non-homogeneous equations of the first degree in x and y;
3. Exact differential equations, and those that can be made exact by the use of integrating factors;
4. Linear equations and equations that are reducible to the linear form.

8. Equations of the form $f_1(x)\,dx + f_2(y)\,dy = 0$. When an equation is in the form

$$f_1(x)\,dx + f_2(y)\,dy = 0,$$

its solution, obtainable at once by integration, is

$$\int f_1(x)\,dx + \int f_2(y)\,dy = c.$$

If the equation is not in the above form, sometimes one can see at a glance how to put it in that form, or, as it is commonly expressed, *to separate the variables.*

Ex. 1. (1) $(1 - x)dy - (1 + y)dx = 0$

can evidently be written

$$(2)\quad \frac{dy}{1+y} - \frac{dx}{1-x} = 0,$$

---

* The student who is proceeding to find the methods of solving differential equations with no more knowledge of the subject than that imparted in the preceding pages, is reminded that he does this, *assuming* (1) that every differential equation with one independent variable has a solution, and (2) that this solution contains a number of arbitrary constants equal to the number indicating its order.

whence, on integrating,

$$(3)\quad \log(1+y)+\log(1-x)=c,$$

and hence

$$(4)\quad (1+y)(1-x)=e^c=c_1.$$

In equations (3) and (4) appear two ways of expressing a general solution of the same equation. Both are equally correct and equally general, but the one has the advantage over the other in neatness and simplicity, and this would make it more serviceable in applications. In some of the examples set, the reduction of the solutions to forms neater and simpler than those which at first present themselves, may require as much labour as the solving of the equations. The solution (4) could have been obtained without separating the variables, if one had noticed that $(1-x)dy-(1+y)dx$ is the differential of $(1-x)(1+y)$. Here, as in the calculus and other subjects, the experience that comes from practice, is the best teacher for showing how to work in the easiest way. Equation (1) can also be put in the form

$$dy-dx-(x\,dy+y\,dx)=0,$$

and another form of the solution obtained, namely,

$$y-x-xy=c_2.$$

Solution (4) reduces to this form on putting c_2 for c_1-1.

Ex. 2. Solve $\dfrac{dy}{dx}+\sqrt{\dfrac{1-y^2}{1-x^2}}=0.$

Ex. 3. Solve $\left(y-x\dfrac{dy}{dx}\right)=a\left(y^2+\dfrac{dy}{dx}\right).$

Ex. 4. Solve $3e^x\tan y\,dx+(1-e^x)\sec^2 y\,dy=0.$

9. Equations homogeneous in x and y. These equations can be put in the form

$$\frac{dy}{dx}=\frac{f_1(x,y)}{f_2(x,y)},$$

where f_1, f_2, are expressions homogeneous and of the same degree in x and y. On putting

$$y=vx,$$

this equation becomes

$$v+x\frac{dv}{dx}=F(v),$$

since each term in f_1, f_2, is of the same degree, say n, in x; and x^n is thus a factor common to both numerator and denominator of its right-hand member.

Separation of the variables gives

$$\frac{dv}{F(v)-v}=\frac{dx}{x},$$

the solution of which gives the relation between x and v, that is, between y and $\frac{y}{x}$, which satisfies the original equation.

Ex. 1. Solve $(x^2+y^2)dx-2\,xy\,dy=0$.

Putting $y=vx$ gives $(1+v^2)dx-2\,v(x\,dv+v\,dx)=0$, which, on separation of the variables, reduces to

$$\frac{dx}{x}-\frac{2\,v}{1-v^2}dv=0.$$

Integrating, $\log x(1-v^2)=\log c$.

On changing the logarithmic form to the exponential, and putting $\frac{y}{x}$ for v, the solution becomes

$$x^2-y^2=cx.$$

Ex. 2. Solve $y^2\,dx+(xy+x^2)dy=0$.

Ex. 3. Solve $x^2y\,dx-(x^3+y^3)dy=0$.

Ex. 4. Solve $(4\,y+3\,x)\frac{dy}{dx}+y-2\,x=0$.

10. Non-homogeneous equations of the first degree in x and y. These equations are of the form

$$\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}. \qquad (1)$$

For x put $x'+h$, and for y put $y'+k$, where h and k are constants; then $dx=dx'$ and $dy=dy'$, and (1) becomes

$$\frac{dy'}{dx'}=\frac{ax'+by'+ah+bk+c}{a'x'+b'y'+a'h+b'k+c'}.$$

If h and k are determined, so that

$$ah+bk+c=0,$$

and $$a'h + b'k + c' = 0,$$

then (1) becomes $$\frac{dy'}{dx'} = \frac{ax' + by'}{a'x' + b'y'}, \tag{2}$$

which is homogeneous in x' and y', and therefore solvable by the method of Art. 9.

If (2) has for its solution

$$f(x', y') = 0,$$

the solution of (1) is $f\{(x-h), (y-k)\} = 0$.

This method fails when $a : b = a' : b'$, h and k then being infinite or indeterminate. Suppose

$$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{m},$$

then (1) can be written

$$\frac{dy}{dx} = \frac{ax + by + c}{m(ax + by) + c'}.$$

On putting v for $ax + by$, the latter equation becomes

$$\frac{dv}{dx} = a + b\frac{v + c}{mv + c'},$$

where the variables can be separated.

Ex. 1. Solve $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0$.

Ex. 2. Solve $(y - 3x + 3)\frac{dy}{dx} = 2y - x - 4$.

11. Exact differential equations. A differential equation which has been formed from its primitive by differentiation, and without any further operation of elimination or reduction, is said to be *exact;* or, in other words, an *exact differential equation* is formed by equating an exact differential to zero. There has now to be found the condition which the coefficients of an equation must satisfy, in order that it may be exact, and also the method of solution to be employed when that condition is satisfied. The question of how to proceed when the condition is not satisfied will be considered next in order.

12. Condition that an equation of the first order be exact. What is the condition that

$$Mdx + Ndy = 0 \tag{1}$$

be an exact differential equation, that is, that $Mdx + Ndy$ be an exact differential? In order that $Mdx + Ndy$ be an exact differential, it must have been derived by differentiating some function u of x and y, and performing no other operation. That is,

$$du = Mdx + Ndy.$$

But
$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Hence, the conditions *necessary*, that $Mdx + Ndy$ be the differential of a function u, are that

$$M = \frac{\partial u}{\partial x}, \text{ and } N = \frac{\partial u}{\partial y}. \tag{2}$$

The elimination of u imposes on M, N, a single condition,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \tag{3}$$

since each of these derivatives is equal to $\dfrac{\partial^2 u}{\partial x\, \partial y}$.

This condition is also *sufficient* for the existence of a function that satisfies (1).* If there is a function u, whose differential du is such that

$$du = Mdx + Ndy,$$

then on integrating relatively to x, since the partial differential Mdx can have been derived only from the terms containing x,

$$u = \int Mdx + \text{terms not containing } x,$$

that is,
$$u = \int Mdx + F(y). \tag{4}$$

---

* For another proof see Note E.

Differentiating both sides of (4) with respect to y,

$$\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\int Mdx+\frac{dF(y)}{dy}.$$

But by (2), $\frac{\partial u}{\partial y}$ must equal N, hence

$$\frac{dF(y)}{dy}=N-\frac{\partial}{\partial y}\int Mdx. \tag{5}$$

The first member of (5) is independent of x; so, also, is the second; for differentiating it with respect to x gives $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$, which, by condition (3), is zero. Integration of both sides of (5) with respect to y gives

$$F(y)=\int\left\{N-\frac{\partial}{\partial y}\int Mdx\right\}dy+a,$$

where a is the arbitrary constant of integration. Substitution in (4) gives

$$u=\int Mdx+\int\left\{N-\frac{\partial}{\partial y}\int Mdx\right\}dy+a.$$

Therefore the primitive of (1), when condition (3) is satisfied, is

$$\int Mdx+\int\left\{N-\frac{\partial}{\partial y}\int Mdx\right\}dy=c. \tag{6}$$

Similarly, $\int Ndy+\int\left\{M-\frac{\partial}{\partial x}\int Ndy\right\}dx=c$

is also a solution.

13. Rule for finding the solution of an exact differential equation. Since all the terms of the solution that contain x must appear in $\int Mdx$, the differential of this integral with respect to y must have all the terms of Ndy that contain x; and therefore (6) can be expressed by the following rule:

To find the solution of an exact differential equation, $Mdx+Ndy=0$, integrate Mdx as if y were constant, integrate the terms in Ndy that do not give terms already obtained, and equate the sum of these integrals to a constant.

Ex. 1. Solve $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0.$

Here, $\dfrac{\partial M}{\partial y} = -4x - 4y = \dfrac{\partial N}{\partial x}$; hence it is an exact equation.

$\int Mdx$ is $\dfrac{x^3}{3} - 2x^2y - 2xy^2$; y^2dy is the only term in Ndy free from x. Therefore the solution is

$$\frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c_1,$$

or

$$x^3 - 6x^2y - 6xy^2 + y^3 = c.$$

The application of the test and of the rule can sometimes be simplified. By picking out the terms of $Mdx + Ndy$ that obviously form an exact differential, or by observing whether any of the terms can take the form $f(u)du$, an expression less cumbersome than the original remains to be tested and integrated.

For instance, the terms of the equation in this example can be rearranged thus:

$$x^2\,dx + y^2\,dy - (4xy + 2y^2)dx - (4xy + 2x^2)dy = 0.$$

The first two terms are exact differentials, and the test has to be applied to the last two only.

Ex. 2.
$$x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2 + y^2} = 0$$

becomes, on dividing the numerator and denominator of the last term by x^2,

$$x\,dx + y\,dy + \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} = 0,$$

each term of which is an exact differential. Integrating,

$$\frac{x^2 + y^2}{2} + \tan^{-1}\frac{y}{x} = c.$$

Ex. 3. Solve $(a^2 - 2xy - y^2)dx - (x + y)^2dy = 0.$

Ex. 4. Solve $(2ax + by + g)dx + (2cy + bx + e)dy = 0.$

Ex. 5. Solve $(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})dy$
$+ (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)dx = 0.$

14. Integrating factors. The differential equation

$$y\,dx - x\,dy = 0$$

is not exact, but when multiplied by $\frac{1}{y^2}$, it becomes

$$\frac{y\,dx - x\,dy}{y^2} = 0,$$

which is exact, and has for its solution

$$\frac{x}{y} = c.$$

When multiplied by $\frac{1}{xy}$, the above equation becomes

$$\frac{dx}{x} - \frac{dy}{y} = 0,$$

which is exact, and has for its solution

$$\log x - \log y = c,$$

which is transformable into the solution first found. Another factor that can be used with like effect on the same equation is $\frac{1}{x^2}$.

Any factor μ, such as $\frac{1}{y^2}$, $\frac{1}{xy}$, $\frac{1}{x^2}$, used above, which changes an equation into an exact differential equation, is called *an integrating factor.*

15. The number of integrating factors is infinite. The number of integrating factors for an equation $Mdx + Ndy = 0$, is infinite. For suppose μ is an integrating factor, then

$$\mu(Mdx + Ndy) = du,$$

and thus $u = c$ is a solution.

Multiplication of both sides by any function of u, say $f(u)$, gives

$$\mu f(u)(Mdx + Ndy) = f(u)\,du;$$

but the second member of the last equation is an exact differential; therefore the first is also, and hence $\mu f(u)$ is an integrating factor of the equation

$$Mdx + Ndy = 0;$$

and as $f(u)$ is an arbitrary function of u, the number of integrating factors is infinite. This fact is, however, of no special assistance in solving the equation.

16. Integrating factors found by inspection. Sometimes integrating factors can be seen at a glance, as in the example of Art. 14.

Ex. 1. Solve $y\,dx - x\,dy + \log x\,dx = 0$.

Here $\log x\,dx$ is an exact differential, and a factor is needed for $y\,dx - x\,dy$. Obviously $\frac{1}{x^2}$ is the factor to be employed, as it will not affect the third term injuriously, from the point of view of integration. The exact equation is then

$$\frac{y\,dx - x\,dy}{x^2} + \frac{\log x}{x^2}dx = 0,$$

the solution of which reduces to

$$cx + y + \log x + 1 = 0.$$

Ex. 2. Solve $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$.

Rearranging the terms, $y\,dx + x\,dy + xy^2\,dx - x^2y\,dy = 0,$

that is, $d(xy) + xy^2\,dx - x^2y\,dy = 0.$

For this, the factor $\frac{1}{x^2y^2}$ immediately suggests itself, and the equation becomes

$$\frac{d(xy)}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0.$$

Integrating, $-\frac{1}{xy} + \log\frac{x}{y} = c_1,$

and transforming, $x = cye^{\frac{1}{xy}}.$

It will be well to try to find an integrating factor by inspection, before having recourse to the rules given in Arts. 17, 18, 19.

Ex. 3. $a(x\,dy + 2\,y\,dx) = xy\,dy.$

Ex. 4. $(x^4e^x - 2\,mxy^2)\,dx + 2\,mx^2y\,dy = 0.$

Ex. 5. $y(2\,xy + e^x)\,dx - e^x dy = 0.$

17. Rules for finding integrating factors. Rules I. and II.

Rules for finding integrating factors in a few cases will now be given.*

Rule I. When $Mx + Ny$ is not equal to zero, and the equation is homogeneous, $\dfrac{1}{Mx+Ny}$ is an integrating factor of

$$Mdx + Ndy = 0.$$

Rule II. When $Mx - Ny$ is not equal to zero, and the equation has the form

$$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0,$$

$\dfrac{1}{Mx-Ny}$ is an integrating factor.

Proof :

$$Mdx + Ndy = \tfrac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

is an identity. This may be written,

$$(a)\quad Mdx + Ndy = \tfrac{1}{2}\left\{(Mx+Ny)d\cdot\log xy + (Mx-Ny)d\cdot\log\frac{x}{y}\right\}.$$

Division of (a) by $Mx + Ny$ gives

$$\frac{Mdx+Ndy}{Mx+Ny} = \tfrac{1}{2}\,d\cdot\log xy + \tfrac{1}{2}\,\frac{Mx-Ny}{Mx+Ny}\,d\cdot\log\frac{x}{y}.$$

Now if $Mdx + Ndy$ is a homogeneous expression, $\dfrac{Mx-Ny}{Mx+Ny}$ is homogeneous and equal to a function of $\dfrac{x}{y}$, and

$$\frac{Mdx+Ndy}{Mx+Ny} = \tfrac{1}{2}\,d\cdot\log xy + \tfrac{1}{2}f\left(\frac{x}{y}\right)d\cdot\log\frac{x}{y},$$

or, since $\dfrac{x}{y} = e^{\log\frac{x}{y}},$

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* For a discussion on and determination of integrating factors, see George Boole, *Differential Equations*, pp. 55–90.

$$\frac{Mdx+Ndy}{Mx+Ny}=\tfrac{1}{2}\,d\cdot\log xy+\tfrac{1}{2}\,F\left(\log\frac{x}{y}\right)d\cdot\log\frac{x}{y},$$

which is an exact differential.

On dividing (a) by $Mx-Ny$, it becomes,

$$\frac{Mdx+Ndy}{Mx-Ny}=\tfrac{1}{2}\,\frac{Mx+Ny}{Mx-Ny}\,d\cdot\log xy+\tfrac{1}{2}\,d\cdot\log\frac{x}{y},$$

and if $Mdx+Ndy$ is of the form $f_1(xy)y\,dx+f_2(xy)x\,dy$, this will be

$$\begin{aligned}\frac{Mdx+Ndy}{Mx-Ny}&=\tfrac{1}{2}\,\frac{f_1(xy)xy+f_2(xy)xy}{f_1(xy)xy-f_2(xy)xy}\,d\cdot\log xy+\tfrac{1}{2}\,d\cdot\log\frac{x}{y},\\&=F_1(xy)d\cdot\log xy+\tfrac{1}{2}\,d\cdot\log\frac{x}{y},\\&=F_2(\log xy)d\cdot\log xy+\tfrac{1}{2}\,d\cdot\log\frac{x}{y},\end{aligned}$$

which is an exact differential.

When $Mx+Ny=0$, $\frac{M}{N}=-\frac{y}{x}$. Substitution for $\frac{M}{N}$ in

$$Mdx+Ndy=0$$

and integration gives the solution $x=cy$.

When $Mx-Ny=0$, $\frac{M}{N}=\frac{y}{x}$. Substitution for $\frac{M}{N}$ in the differential equation and integration gives the solution $xy=c$.

Ex. 1. Solve $(x^2y-2\,xy^2)dx-(x^3-3\,x^2y)dy=0$.

Ex. 2. Solve Ex. 3, Art. 9, by this method.

Ex. 3. Solve $y(xy+2\,x^2y^2)dx+x(xy-x^2y^2)dy=0$.

18. Rules III. and IV.

RULE III. When $\dfrac{\dfrac{dM}{dy}-\dfrac{dN}{dx}}{N}$ is a function of x alone, say $f(x)$, $e^{\int f(x)dx}$ is an integrating factor.

For, multiplication of $Mdx + Ndy = 0$ by that factor gives, say, $M_1dx + N_1dy = 0$; and differentiation will show that

$$\frac{dM_1}{dy} = \frac{dN_1}{dx}.$$

Ex. **1.** $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0.$

Ex. **2.** $(x^2 + y^2)dx - 2xy\,dy = 0.$

Rule IV. When $\dfrac{\dfrac{dN}{dx} - \dfrac{dM}{dy}}{M}$ is a function of y alone, say $F(y)$, $e^{\int F(y)dy}$ is an integrating factor.

This can be shown in the same way as in the preceding rule.

Ex. **3.** Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$

Ex. **4.** Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0.$

19.[*] **Rule V.** $x^{\kappa m-1-\alpha}y^{\kappa n-1-\beta}$, where κ has any value, is an integrating factor of

$$x^{\alpha}y^{\beta}(my\,dx + nx\,dy) = 0,$$

for on using the factor, the equation becomes

$$\frac{1}{\kappa}\,d(x^{\kappa m}y^{\kappa n}) = 0.$$

Moreover, when an equation can be put in the form

$$x^{\alpha}y^{\beta}(my\,dx + nx\,dy) + x^{\alpha_1}y^{\beta_1}(m_1y\,dx + n_1x\,dy) = 0,$$

an integrating factor can be easily obtained. A factor that will make $x^{\alpha}y^{\beta}(my\,dx + nx\,dy)$ an exact differential is $x^{\kappa m-1-\alpha}y^{\kappa n-1-\beta}$, where κ has any value; and a factor that will make

$$x^{\alpha_1}y^{\beta_1}(m_1y\,dx + n_1x\,dy)$$

an exact differential is $x^{\kappa_1 m_1-1-\alpha_1}y^{\kappa_1 n_1-1-\beta_1}$, where κ_1 has any value.

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[*] See L'Abbé Moigno, *Calcul Différentiel et Intégral* (published 1844), t. II., No. 147, p. 355; Johnson, *Differential Equations*, Art. 32.

These two factors are identical if

$$\kappa m - 1 - \alpha = \kappa_1 m_1 - 1 - \alpha_1,$$

and
$$\kappa n - 1 - \beta = \kappa_1 n_1 - 1 - \beta_1.$$

Values of κ and κ_1 can be found to satisfy these conditions.

Ex. 1. Solve $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$.

Rearranging in the form above,

$$y^2(y\,dx + 2x\,dy) - x^2(2y\,dx + x\,dy) = 0.$$

For the first term $\alpha = 0$, $\beta = 2$, $m = 1$, $n = 2$, and hence $x^{\kappa-1}y^{2\kappa-1-2}$ is its integrating factor. For the second term $\alpha = 2$, $\beta = 0$, $m = 2$, $n = 1$, and hence $x^{2\kappa'-1-2}y^{\kappa'-1}$ is its integrating factor.

These factors are the same if

$$\kappa - 1 = 2\kappa' - 1 - 2,$$

and
$$2\kappa - 1 - 2 = \kappa' - 1.$$

On solving for κ and κ', $\kappa = 2 = \kappa'$, and therefore xy is the common integrating factor for both terms.

The equation when made exact is

$$xy\{y^2(y\,dx + 2x\,dy) - x^2(2y\,dx + x\,dy)\} = 0.$$

$$\therefore \frac{x^2y^4}{2} - \frac{x^4y^2}{2} = c, \text{ or } x^2y^2(y^2 - x^2) = c.$$

Ex. 2. Solve $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$.

Ex. 3. Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$.

20. Linear equations. A differential equation is said to be linear when the dependent variable and its derivatives appear only in the first degree. The form of the linear equation of the first order is

$$\frac{dy}{dx} + Py = Q, \qquad (1)$$

where P and Q are functions of x or constants.

The solution of
$$\frac{dy}{dx} + Py = 0,$$

that is, of
$$\frac{dy}{y} = -P dx,$$

is
$$y = ce^{-\int P dx}, \text{ or } ye^{\int P dx} = c.$$

On differentiation the latter form gives

$$e^{\int P dx}(dy + Py\,dx) = 0,$$

which shows that $e^{\int P dx}$ is an integrating factor of (1).

Multiplication of (1) by that factor changes it into the exact equation,

$$e^{\int P dx}(dy + Py\,dx) = e^{\int P dx}Q dx,$$

which on integration gives

$$ye^{\int P dx} = \int e^{\int P dx} Q dx + c,$$

or
$$y = e^{-\int P dx}\left\{\int e^{\int P dx} Q dx + c\right\}. \tag{2}$$

The latter can be used as a formula for obtaining the value of y in a linear equation of the form (1).* The student is advised to make himself familiar with the linear equation and its solution, since it appears very frequently.

Ex. 1. Solve $x\dfrac{dy}{dx} - ay = x + 1$.

This is linear since it is of the first degree in y and $\dfrac{dy}{dx}$. Putting it in the regular form, it becomes

$$\frac{dy}{dx} - \frac{a}{x}y = \frac{x+1}{x}.$$

Here $P = -\dfrac{a}{x}$, and the integrating factor $e^{\int P dx}$ is $\dfrac{1}{x^a}$.

Using that factor, the equation changes to

$$\frac{1}{x^a}dy - \frac{ay}{x^{a+1}}dx = \frac{x+1}{x^{a+1}}dx.$$

$$\therefore \frac{y}{x^a} = \int \frac{x+1}{x^{a+1}}dx + c,$$

whence
$$y = \frac{x}{1-a} - \frac{1}{a} + cx^a.$$

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* Gottfried Wilhelm Leibniz (1646–1716), who, it is generally admitted, invented the differential calculus independently of Newton, appears to have been the first who obtained the solution (2).

The values of P and Q might have been substituted in the value of y as expressed in (2).

Ex. 2. Solve $\dfrac{dy}{dx} + y = e^{-x}$.

Ex. 3. Solve $\cos^2 x \dfrac{dy}{dx} + y = \tan x$.

Ex. 4. Solve $(x+1)\dfrac{dy}{dx} - ny = e^x(x+1)^{n+1}$.

Ex. 5. Solve $(x^2+1)\dfrac{dy}{dx} + 2xy = 4x^2$.

21. Equations reducible to the linear form. Sometimes equations not linear can be reduced to the linear form. In particular, this is the case with those of the form*

$$\frac{dy}{dx} + Py = Qy^n, \tag{1}$$

where P and Q are functions of x. For, on dividing by y^n and multiplying by $(-n+1)$, this equation becomes

$$(-n+1)y^{-n}\frac{dy}{dx} + (-n+1)Py^{-n+1} = (-n+1)Q;$$

on putting v for y^{-n+1}, it reduces to

$$\frac{dv}{dx} + (1-n)Pv = (1-n)Q,$$

which is linear in v.

Ex. 1. Solve $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2y^6$.

Division by y^6 gives $\qquad y^{-6}\dfrac{dy}{dx} + \dfrac{y^{-5}}{x} = x^2$.

On putting v for y^{-5}, this reduces to $\dfrac{dv}{dx} - \dfrac{5}{x}v = -5x^2$, the linear form.

Its solution is $\qquad v = y^{-5} = cx^5 + \dfrac{5x^3}{2}$.

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* This is also called *Bernoulli's equation*, after James Bernoulli (1654–1705), who studied it in 1695.

NOTE. In general, an equation of the form

$$f'(y)\frac{dy}{dx} + Pf(y) = Q,$$

where P and Q are functions of x, on the substitution of v for $f(y)$ becomes

$$\frac{dv}{dx} + Pv = Q,$$

which is linear.

Ex. 2. Solve $(1 + y^2)dx = (\tan^{-1} y - x)dy$.

This can be put in the form

$$\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1} y}{1+y^2},$$

which is a linear equation, y being taken as the independent variable. Integration as in the last article gives the solution

$$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

Ex. 3. Solve $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x^2y^{\frac{4}{3}}$.

Ex. 4. Solve $\dfrac{dy}{dx} + \dfrac{xy}{1-x^2} = xy^{\frac{1}{2}}$.

Ex. 5. Solve $3x(1 - x^2)y^2\dfrac{dy}{dx} + (2x^2 - 1)y^3 = ax^3$.

EXAMPLES ON CHAPTER II.

Equations can sometimes be reduced to standard forms by substitutions.

1. $(x+y)^2\dfrac{dy}{dx} = a^2$. [Put $x+y=v$.]

2. $x\dfrac{dy}{dx} - y = \sqrt{x^2+y^2}$.

3. $x\dfrac{dy}{dx} - y = x\sqrt{x^2+y^2}$.

4. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$.

5. $(x^2 - yx^2)\dfrac{dy}{dx} + y^2 + xy^2 = 0$.

6. $\dfrac{dy}{dx} + \dfrac{1-2x}{x^2}y = 1$.

7. $3\dfrac{dy}{dx} + \dfrac{2}{x+1}y = \dfrac{x^3}{y^2}$.

8. $(2x - y + 1)dx + (2y - x - 1)dy = 0$.

9. $\dfrac{dy}{dx} + \dfrac{y}{(1-x^2)^{\frac{3}{2}}} = \dfrac{x+\sqrt{1-x^2}}{(1-x^2)^2}$.

10. $x\dfrac{dy}{dx} + \dfrac{y^2}{x} = y$.

11. $(x^2 + y^2 - a^2)x\,dx + (x^2 - y^2 - b^2)y\,dy = 0$.

12. $\frac{dy}{dx}+\frac{4x}{x^2+1}y=\frac{1}{(x^2+1)^3}$.

13. $x^2y\,dx-(x^3+y^3)dy=0$.

14. $x(1-x^2)\frac{dy}{dx}+(2x^2-1)y=ax^3$.

15. $(x^2+y^2+1)dx-2xy\,dy=0$.

16. $x\,dx+y\,dy=m(x\,dy-y\,dx)$.

17. Integrate Ex. 16, after changing the variables by the transformation $x=r\cos\theta,\ y=r\sin\theta$.

18. $\left(1+e^{\frac{x}{y}}\right)dx+e^{\frac{x}{y}}\left(1-\frac{x}{y}\right)dy=0$.

19. $\frac{dy}{dx}+y\cos x=y^n\sin 2x$.

20. $(x+1)\frac{dy}{dx}+1=2e^{-y}$.

21. $\frac{dy}{dx}=x^3y^3-xy$.

22. $y\,dx+(ax^2y^n-2x)dy=0$.

23. $(1+6y^2-3x^2y)\frac{dy}{dx}=3xy^2-x^2$.

24. $y(x^2+y^2+a^2)\frac{dy}{dx}+x(x^2+y^2-a^2)=0$.

25. $(x^2y^3+xy)dy=dx$.

26. $y\frac{dy}{dx}=ax$.

27. $\sqrt{a^2+x^2}\frac{dy}{dx}+y=\sqrt{a^2+x^2}-x$.

28. $(x+y)\frac{dy}{dx}+(x-y)=0$.

29. $y\,dy+by^2dx=a\cos x\,dx$.

30. $2xy\,dx+(y^2-x^2)dy=0$.

31. $\left(xy^2-e^{\frac{1}{x^3}}\right)dx-x^2y\,dy=0$.

32. $y-x\frac{dy}{dx}=b\left(1+x^2\frac{dy}{dx}\right)$.

33. $(3y+2x+4)dx-(4x+6y+5)dy=0$.

34. $(x^3y^3+x^2y^2+xy+1)y+(x^3y^3-x^2y^2-xy+1)x\frac{dy}{dx}=0$.

35. $(2x^2y^2+y)dx-(x^3y-3x)dy=0$.

36. $y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$.

37. $\frac{dy}{dx}+\frac{n}{x}y=\frac{a}{x^n}$.

38. $(x-y)^2\frac{dy}{dx}=a^2$.

CHAPTER III.

EQUATIONS OF THE FIRST ORDER, BUT NOT OF THE FIRST DEGREE.

22. Equations that can be resolved into component equations of the first degree. In what follows, $\frac{dy}{dx}$ will be denoted by p.

The type of the equation of the first order and nth degree is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_{n-1} p + P_n = 0, \qquad (1)$$

where $P_1, P_2, \cdots, P_n$, are functions of x and y.

Two cases appear for consideration, viz.:

(a) where the first member of (1) can be resolved into rational factors of the first degree;

(b) where that member cannot be thus factored.

In the first case (1) can take the form

$$(p - R_1)(p - R_2) \cdots (p - R_n) = 0. \qquad (2)$$

Equation (1) is satisfied by a value of y that will make any factor of the first member of (2) equal to zero. Therefore, to obtain the solutions of (1), equate each of the factors in (2) to zero, and obtain the solutions of the n equations thus formed. The n solutions can be left distinct or combined into one.

Suppose the solutions derived for (2) are

$$f_1(x, y, c_1) = 0, f_2(x, y, c_2) = 0, \cdots, f_n(x, y, c_n) = 0,$$

where $c_1, c_2, \cdots, c_n$, are the arbitrary constants of integration.

These solutions are evidently just as general, if $c_1 = c_2 = \cdots = c_n$, since all the c's can have any one of an infinite number of values; and the solutions will then be

$$f_1(x, y, c) = 0,\ f_2(x, y, c) = 0, \cdots, f_n(x, y, c) = 0.$$

These can be combined into one equation; namely,

$$f_1(x, y, c) f_2(x, y, c) \cdots f_n(x, y, c) = 0. \tag{3}$$

Ex. 1. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$, can be written

$$p(p + 2x)(p - y^2) = 0.$$

Its component equations are

$$p = 0, \quad p + 2x = 0, \quad p - y^2 = 0,$$

of which the solutions are

$$y = c, \quad y + x^2 = c, \quad \text{and } xy + cy + 1 = 0,$$

respectively. The combined solution is

$$(y - c)(y + x^2 - c)(xy + cy + 1) = 0.$$

When the equation in p is of the second degree, sometimes the solution readily presents itself in the form (3) as in the next example.

Ex. 2. Solve $\left(\frac{dy}{dx}\right)^2 - ax^3 = 0.$

$$\frac{dy}{dx} = \pm a^{\frac{1}{2}}x^{\frac{3}{2}}.$$

Integrating, $$y + c = \pm \tfrac{2}{5} a^{\frac{1}{2}}x^{\frac{5}{2}}.$$

Rationalizing, $$25(y + c)^2 = 4ax^5,$$

or $$25(y + c)^2 - 4ax^5 = 0.$$

Ex. 3. Solve $p^3(x + 2y) + 3p^2(x + y) + (y + 2x)p = 0.$

Ex. 4. Solve $\left(\frac{dy}{dx}\right)^3 = ax^4.$

Ex. 5. Solve $4y^2p^2 + 2pxy(3x + 1) + 3x^3 = 0.$

Ex. 6. Solve $p^2 - 7p + 12 = 0.$

23. Equations that cannot be resolved into component equations. Methods which may be tried for solving equation (1) of the last article, when its first member cannot be resolved into rational linear factors, (case (*b*) Art. 22), will now be shown.

That equation, which may be expressed in the form

$$f(x, y, p) = 0,$$

may have one or more of the following properties.

(*a*) It may be solvable for y.
(*b*) It may be solvable for x.

The case where it is solvable for p has been considered in the preceding section.

(*c*) It either may not contain x, or it may not contain y.
(*d*) It may be homogeneous in x and y.
(*e*) It may be of the first degree in x and y.

24. Equations solvable for y. When the condition (*a*) holds, $f(x, y, p) = 0$ can be put in the form

$$y = F(x, p).$$

Differentiation with respect to x gives

$$p = \phi\left(x, p, \frac{dp}{dx}\right),$$

which is an equation in two variables x and p; from this it may be possible to deduce a relation

$$\psi(x, p, c) = 0.$$

The elimination of p between the latter and the original equation gives a relation involving x, y, and c, which is the solution required.

When the elimination of p between these equations is not easily practicable, the values of x and y in terms of p as a parameter can be found, and these together will constitute the solution.

Ex. 1. Solve $x - yp = ap^2$.

Here $$y = \frac{x - ap^2}{p}.$$

Differentiating and clearing of fractions,

$$(ap^2 + x)\frac{dp}{dx} = p(1 - p^2).$$

This can be put in the linear form

$$\frac{dx}{dp} - \frac{1}{p(1 - p^2)}x = \frac{ap}{1 - p^2}.$$

Solving, $$x = \frac{p}{\sqrt{1-p^2}}(c + a\sin^{-1}p).$$

Substituting in the value for y above,

$$y = -ap + \frac{1}{\sqrt{1-p^2}}(c + a\sin^{-1}p).$$

Ex. 2. Solve $y = x + a\tan^{-1}p$.

Ex. 3. Solve $4y = x^2 + p^2$.

Ex. 4. Solve $xp^2 - 2yp + ax = 0$.

25. Equations solvable for x. When condition (*b*) holds, $f(x, y, p) = 0$ can be put in the form

$$x = F(y, p).$$

Differentiation with respect to y gives

$$\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right),$$

from which a relation between p and y may possibly be obtained, say,

$$f(y, p, c) = 0.$$

Between this and the given equation p may be eliminated, or x and y expressed in terms of p as in the last article.

Ex. 1. Solve $x = y + p^2$.

Ex. 2. Solve $x = y + a\log p$.

Ex. 3. Solve $p^2y + 2px = y$.

26. Equations that do not contain x; that do not contain y. When the equation has the form

$$f(y, p) = 0,$$

and this is solvable for p, it will give

$$\frac{dy}{dx} = \phi(y),$$

which is integrable.

If it is solvable for y, it will give

$$y = F(p),$$

which is the case of Art. 24.

When the equation is of the form

$$f(x, p) = 0,$$

and this is solvable for p, it will give

$$\frac{dy}{dx} = \phi(x),$$

which is immediately integrable.

If it is solvable for x, it will give

$$x = F(p),$$

which is the case of Art. 25.

It is to be noticed that in equations having either of the properties (c) Art. 23 and not solvable for p, on solving for x or y the differentiation is made with respect to the absent variable.

By differentiating in cases (a), (b), (c), there is a chance of obtaining a differential equation, by means of which another relation may be found between p and x or y in addition to the original relation. These two relations will then serve either for the elimination of p, or for the expression of x and y in terms of p.

Ex. **1.** Solve $y = 2p + 3p^2$.

Ex. **2.** Solve $x(1 + p^2) = 1$.

Ex. **3.** Solve $x^2 = a^2(1 + p^2)$.

Ex. **4.** Solve $y^2 = a^2(1 + p^2)$.

27. Equations homogeneous in x and y. When the equation is homogeneous in x and y, it can be put in the form

$$F\left(\frac{dy}{dx}, \frac{y}{x}\right) = 0.$$

It may be possible to solve this for $\frac{dy}{dx}$, and then to proceed as in Art. 9; or to solve it for $\frac{y}{x}$, and obtain

$$y = xf(p),$$

which comes under case (a) Art. 23.

Proceeding as in Art. 24, differentiate with respect to x,

$$p = f(p) + xf'(p)\frac{dp}{dx};$$

whence

$$\frac{dx}{x} = \frac{f'(p)\,dp}{p - f(p)},$$

where the variables are separated.

Ex. 1. Solve $y^2 + xyp - x^2p^2 = 0$.

Ex. 2. Solve $y = yp^2 + 2px$.

28. Equations of the first degree in x and y. Clairaut's equation. When the condition (e) Art. 23, holds, the equation, being solvable for x, and for y as well, comes under cases (a) and (b) considered in Arts. 24, 25. However, there is one particular form of these equations of the first degree in x and y that is of special importance, namely,

$$y = px + f(p),$$

which is known as *Clairaut's equation.*[*]

Differentiation with respect to x gives

$$p = p + \{x + f'(p)\}\frac{dp}{dx},$$

whence

$$x + f'(p) = 0,$$

or

$$\frac{dp}{dx} = 0.$$

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[*] Alexis Claude Clairaut (1713–1765), celebrated for his researches on the figure of the earth and on the motions of the moon, was the first who had the idea of aiding the integration of differential equations by differentiating them. He applied it to the equation that now bears his name, and published the method in 1734.

From the latter equation, it follows that $p = c$, and hence

$$y = cx + f(c)$$

is the solution.

The equation $x + f'(p) = 0$ is considered in Art. 34.

Any equation satisfying condition (e) can be put in the form

$$y = xf_1(p) + f_2(p).$$

If $f_1(p) = p$, it is in Clairaut's form. By proceeding as in Art. 24 and differentiating with respect to x there is obtained

$$p = f_1(p) + \{xf_1'(p) + f_2'(p)\}\frac{dp}{dx}.$$

$$\therefore\ \frac{dx}{dp} = \frac{f_1'(p)}{p - f_1(p)}x + \frac{f_2'(p)}{p - f_1(p)},$$

which is linear in x; and from this a relation between x and p may be deduced.

The student should be familiar enough with Clairaut's form to recognize it readily.

Some equations are reducible to this form; Ex. 2 is an illustration.

Ex. 1. Solve $y = (1 + p)x + p^2$.

Differentiating, $$p = 1 + p + (x + 2p)\frac{dp}{dx}.$$

$$\therefore\ \frac{dx}{dp} + x = -2p,$$

which is linear. Solving,

$$x = 2(1 - p) + ce^{-p};$$

and hence $$y = 2 - p^2 + (1 + p)ce^{-p}$$

from the given equation.

Ex. 2. Solve $x^2(y - px) = yp^2$.

On putting $x^2 = u$, and $y^2 = v$, the equation becomes

$$v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2,$$

which is Clairaut's form.

$$\therefore\ v = cu + c^2,$$

and hence $$y^2 = cx^2 + c^2.$$

Ex. 3. Solve $y = xp + \sin^{-1} p$.

Ex. 4. Solve $e^{4x}(p-1) + e^{2y}p^2 = 0$.

Ex. 5. Solve $xy(y - px) = x + py$.

Solving for x or y may be of service in the case of equations of the first degree in p; this is illustrated in Ex. 6.

Ex. 6. Solve $\dfrac{dy}{dx} + 2xy = x^2 + y^2$.

The solution for y gives the equation $\quad y = x + \sqrt{p}$,

which is of the form discussed in Art. 24.

The solution is
$$y = x + \frac{c + e^{2x}}{c - e^{2x}}.$$

29. Summary. What has been said in this chapter concerning the equation $f(x, y, p) = 0$, of degree higher than the first in p, may be thus summed up:

Either solve $f(x, y, p) = 0$ for p, and obtain a solution corresponding to each value of p; or,

Solve for y or x, and, by differentiating with respect to x or y, obtain an equation, whence another relation between p and x or y can be found. This new relation, taken in connection with the original equation, will serve either for the elimination of p, or for the evaluation of x and y in terms of p; the eliminant or the values of x and y will be the solution.

EXAMPLES ON CHAPTER III.

1. $x^2\left(\dfrac{dy}{dx}\right)^2 - 2xy\dfrac{dy}{dx} + 2y^2 - x^2 = 0$.

2. $y = p(x - b) + \dfrac{a}{p}$.

3. $xy^2(p^2 + 2) = 2py^3 + x^3$.

4. $y = -xp + x^4p^2$.

5. $p^2 - 9p + 18 = 0$.

6. $ayp^2 + (2x - b)p - y = 0$.

7. $y - px = \sqrt{1 + p^2}\,\phi(x^2 + y^2)$.

8. $(xp - y)^2 = a(1 + p^2)(x^2 + y^2)^{\frac{3}{2}}$.

9. $(xp - y)^2 = p^2 - 2\dfrac{y}{x}p + 1$.

10. $3p^2y^2 - 2xyp + 4y^2 - x^2 = 0$.

11. $(x^2 + y^2)(1 + p)^2 - 2(x + y)(1 + p)(x + yp) + (x + yp)^2 = 0$.

12. $(py+nx)^2=(y^2+nx^2)(1+p^2).$

13. $y^2(1-p^2)=b.$

14. $(px-y)(py+x)=h^2p.$

15. $p^2+2\,py\cot x=y^2.$

16. $\left(p^2-\dfrac{1}{a^2-x^2}\right)\left(p-\sqrt{\dfrac{y}{x}}\right)=0.$

17. $x+\dfrac{p}{\sqrt{1+p^2}}=a.$

18. $y-2\,px=f(xp^2).$

19. $xyp^2+p(3\,x^2-2\,y^2)-6\,xy=0.$

20. $p^3-4\,xyp+8\,y^2=0.$

21. $p^3-(x^2+xy+y^2)p^2+(x^3y+x^2y^2+xy^3)p-x^3y^3=0.$

22. $p^3+mp^2=a(y+mx).$

23. $e^{3x}(p-1)+p^3e^{2y}=0.$

24. $\left(1-y^2+\dfrac{y^4}{x^2}\right)p^2-2\,\dfrac{y}{x}\,p+\dfrac{y^2}{x^2}=0.$

25. $y-(1+p^2)^{-\frac{1}{2}}=b.$

26. $y=px+\dfrac{m}{p}.$

27. $y=2\,px+y^2p^3.$

CHAPTER IV.

SINGULAR SOLUTIONS.

30. References to algebra and geometry. In this explanation of singular solutions,* use will be made of a few definitions and principles of algebra and geometry; particularly of the discriminant in the one, and of envelopes in the other. Articles 31 and 32 will serve to recall some of them. The student is advised to consult a work on the theory of equations and a differential calculus concerning these points.

31. The discriminant. The discriminant of an equation involving a single variable is the simplest function of the coefficients in a rational integral form, whose vanishing is the condition that the equation have two equal roots. For example, the value of x in $ax^2 + bx + c = 0$ is $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$; and so the condition that the equation have equal roots is that $b^2 - 4ac$ be equal to zero. The discriminant is $b^2 - 4ac$; the equation $b^2 - 4ac = 0$ will be called *the discriminant relation.*

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* Leibniz in 1694 (see footnote, p. 27), Brook Taylor (1685–1731), the discoverer of the theorem called by his name, in 1715, and Clairaut (see footnote, p. 36) were the first to detect singular solutions of differential equations. Clairaut refers to these solutions in a paper published in the *Memoirs of the Paris Academy of Sciences* in 1734. Their geometrical significance was first pointed out by Lagrange (see footnote, p. 155) in an article published in the *Memoirs of the Berlin Academy of Sciences* in 1774, in which he also showed a way of obtaining them. The theory at present accepted is that expounded by Arthur Cayley (1821–1895) in an article in the *Messenger of Mathematics*, Vol. II., 1872.

When the equation is quadratic, the discriminant can be written immediately; but when it is such that the condition for equal roots is not easily perceived, the discriminant is found in the following way. The given equation being $F=0$, form another equation by differentiating F with respect to the variable, and eliminate the variable between the two equations.

For example,

$$\phi(x, y, c)=0$$

may be looked on as an equation in c, its coefficients then being functions of x and y. The simplest rational function of x and y, whose vanishing expresses that the equation $\phi(x, y, c)=0$ has equal roots for c, is called the c discriminant of ϕ, and is obtained by eliminating c between the equations,

$$\phi(x, y, c)=0, \quad \frac{d\phi}{dc}=0.$$

Thus the c discriminant relation represents the locus, for each point of which $\phi(x, y, c)=0$ has equal values of c.

Similarly, the p discriminant of $f(x, y, p)=0$, the differential equation corresponding to $\phi(x, y, c)=0$, is obtained by eliminating p between the equations,

$$f(x, y, p)=0, \quad \frac{df}{dp}=0.$$

Thus the p discriminant relation represents the locus, for each point of which $f(x, y, p)=0$ has equal values of p.

In order that there may be a c and a p discriminant, the above equations must be of the second degree at least in c and p. In Art. 6 it was pointed out that these equations are of the same degree in c and p, and hence, if there is a p discriminant, there must be a c discriminant.

32. The envelope. If in $\phi(x, y, c)=0$, c be given all possible values, there is obtained a set of curves, infinite in number, of the same kind. Suppose that the c's are arranged in order of magnitude, the successive c's thus differing by infinitesimal amounts, and that all these curves are drawn. Curves corresponding to two consecutive values of c are called *consecutive curves*, and their intersection is called *an ultimate point of intersection*. The limiting position of these points of intersection includes *the envelope* of the system of curves. It is shown in works on the differential calculus, that the envelope is part of the locus of the equation obtained by eliminating c between

$$\phi(x, y, c) = 0,$$

and

$$\frac{d\phi}{dc} = 0;$$

that is, the envelope is part of the locus of the c discriminant relation. This might have been anticipated, because in the limit the c's for two consecutive curves become equal, and the c discriminant relation represents the locus of points for which $\phi(x, y, c) = 0$ will have equal values of c.

It is also shown in the differential calculus, that at any point on the envelope, the latter is touched by some curve of the system; that is, that the envelope and some one of the curves have the same value of p at the point.

33. The singular solution. Suppose that

$$f(x, y, p) = 0 \tag{1}$$

is the differential equation, which has

$$\phi(x, y, c) = 0 \tag{2}$$

for its solution. It has been seen, in Arts. 4–6, that the system of curves which is the locus of $f(x, y, p) = 0$ is the set of curves obtained by giving c all possible values in (2). The x, y, p, at each point on the envelope of the system of curves which is the locus of (2), being identical with the x, y, p, of some point on one of these curves, satisfy (1). Therefore the equation of the envelope is also a solution of that differential equation. This is called *the singular solution*. It is distinguished from a particular solution, in that it is not contained in the general solution; that is, it is not derived by giving the constant in the general solution a particular value.

The singular solution may be obtained from the differential equation directly, without any knowledge of the general solution. For, at the points of ultimate intersection of consecutive curves, the p's for the intersecting curves become equal, and thus the locus of the points where the p's have equal roots will include the envelope; that is, the p discriminant relation of (1) contains the equation of the envelope of the system of curves represented by (2). In the next article, it will be shown that the p and c discriminant relations may sometimes represent other loci besides the envelope: that is, they may contain other equations besides the singular solution. The part of these relations that satisfies the differential equation is the singular solution.

Ex. 1. $y = x\frac{dy}{dx} + a\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$,

which is in Clairaut's form, has for its solution

$$y = cx + a\sqrt{1 + c^2}.$$

This, on rationalization, becomes

$$c^2(a^2 - x^2) + 2\,cxy + a^2 - y^2 = 0,$$

and hence the condition for equal roots is

$$x^2 + y^2 = a^2.$$

This relation satisfies the given equation, and hence is the singular solution.

In this example, the general integral represents the system of lines $y = cx + a\sqrt{1 + c^2}$, all of which touch the circle $x^2 + y^2 = a^2$.

Ex. 2. Find the general and the singular solutions of $p^2 + xp - y = 0$.

Ex. 3. Find the general and the singular solutions of $dy\sqrt{x} = dx\sqrt{y}$.

Ex. 4. Find the singular solution of $x^2p^2 - 3\,xyp + 2\,y^2 + x^3 = 0$.

Ex. 5. Find the general and the singular solutions of

$$\left(1 + \frac{dy}{dx}\right)^3 = \frac{27}{8\,a}(x + y)\left(1 - \frac{dy}{dx}\right)^3.$$

34. Clairaut's equation. In finding the solution of Clairaut's form in Art. 28, there appeared the equation

$$x + f'(p) = 0, \tag{3}$$

which is as important as the equation $\frac{dp}{dx} = 0$, that appeared with it. The foregoing shows what part equation (3) plays in solving Clairaut's equation. On differentiating $y = px + f(p)$ with respect to p, (3) is obtained. The elimination of p between these two equations gives the p discriminant relation, which here represents the envelope of the system of lines

$$y = cx + f(c)$$

represented by the general solution.

35. Relations, not solutions, that may appear in the p and c discriminant relations. It has been pointed out that the p discriminant relation of $f(x, y, p) = 0$ represents the locus, for each point of which $f(x, y, p) = 0$ will have equal values of p; and that the c discriminant relation of $\phi(x, y, c) = 0$, the general solution of the former equation, represents the locus for each point of which $\phi(x, y, c) = 0$ will have equal values of c. It is known also that each point on the envelope of the system $\phi(x, y, c) = 0$ is a point of ultimate intersection of a pair of consecutive curves of that system; and, moreover, that at each point on the envelope there will be two equal values of p, one for each of the consecutive curves intersecting at the point; and that, therefore, the singular solution, representing the envelope, must appear in both the p and the c discriminant relations. But the question then arises, may there not be other loci besides the envelope, whose points will make $f(x, y, p) = 0$ give equal values of p, or make $\phi(x, y, c) = 0$ give equal values of c? In other words, while the p and the c discriminant relations must both contain the singular solution, which represents the envelope if there be one, may they not each contain something else?

36. Equation of the tac-locus. At a point satisfying the p discriminant relation there are two equal values of p; these equal p's, however, may belong to two curves of the system that are not consecutive, but which happen to touch at the point in question. Such a point of contact of two non-consecutive curves is on a locus called *the tac-locus* of the system of curves. The equations representing the tac-locus, while thus appearing in the p discriminant relation, will not be contained in that of the c discriminant; since the touching curves, being non-consecutive, will have different c's.

Ex. Examine $y^2(1+p^2)=r^2.$

Solving for p, $p=\dfrac{\sqrt{r^2-y^2}}{y}.$

Integrating and rationalizing,

$$y^2+(x+c)^2=r^2.$$

The general solution, therefore, represents a system of circles having a radius equal to r and their centres on the x axis.

The c discriminant relation is $y^2-r^2=0,$

and that of the p discriminant is $y^2(y^2-r^2)=0.$

Thus the locus of the latter is made up of the loci $y=\pm r$ and of $y=0$ counted twice.

The equations $y=\pm r$, that appear in both the p and the c discriminant relations, satisfy the differential equation, and hence form the singular solution; they represent the envelope.

The equation $y=0$, as is apparent on substitution, does not satisfy the differential equation. Through every point on the locus $y=0$, two circles of the system can be drawn touching each other; that equation, therefore, represents the tac-locus.

The student is advised to make a figure, showing the set of circles, their envelope, and the tac-locus, as it will help him to understand this and the preceding articles.

37. Equation of the nodal locus. The c discriminant relation, like that of the p discriminant, may contain an equation having a locus, the x, y, p, of whose points will not satisfy the differential equation.

The general solution $\phi(x, y, c) = 0$ may represent a set of curves each of which has a double point. Changing the c changes the position of the curve, but not its character. These curves being supposed drawn, the double points will lie on a curve which is called *the nodal locus.* In the limit two consecutive curves of the system will have their nodes in coincidence upon the nodal locus. The node is thus one of the ultimate points of intersection of consecutive curves; and, therefore, the equation of this locus must appear in the c discriminant relation. But in Fig. 1, where A, B, $\cdots$, are the curves and L is the nodal locus, at any point the p for the nodal locus L is different from the p's of the particular curve that passes through the point; and hence the x, y, p, belonging to L at the point, will not satisfy the differential equation.

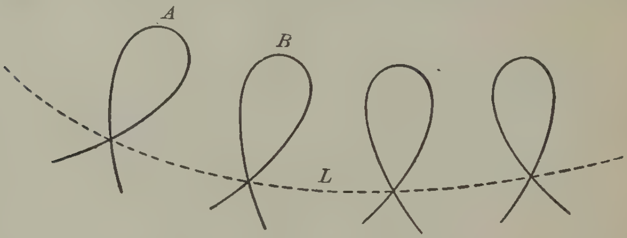

FIG. 1.

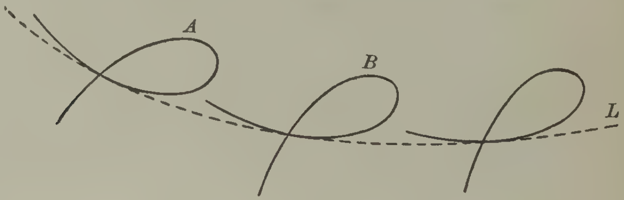

FIG. 2.

And, in general, the x, y, p, at points on the nodal locus will not satisfy the differential equation; for the case would be exceptional where the p at any point on the nodal locus would coincide with a p for a curve of the general solution passing through that point; where, in other words, the nodal locus would also be an envelope, as in Fig. 2, in which $A, B, \cdots, L$, have the same signification as in Fig. 1.

Ex. $xp^2-(x-a)^2=0$ has for its general solution

$$y+c=\tfrac{2}{3}x^{\frac{3}{2}}-2ax^{\frac{1}{2}};$$

that is, $$\tfrac{9}{4}(y+c)^2=x(x-3a)^2.$$

The p discriminant relation is $x(x-a)^2=0$,

and that of the c discriminant, $x(x-3a)^2=0$.

The relation $x=0$ satisfies the differential equation; hence it is the singular solution and represents the envelope locus.

The relation $x-a=0$, which appears only in the p discriminant, does not satisfy the differential equation; it represents the tac-locus. And $x-3a=0$, which is in the c discriminant, does not satisfy the original equation; it represents the nodal locus.

Figure 3 shows some of the curves of the system, the envelope, the tac, and the nodal loci.

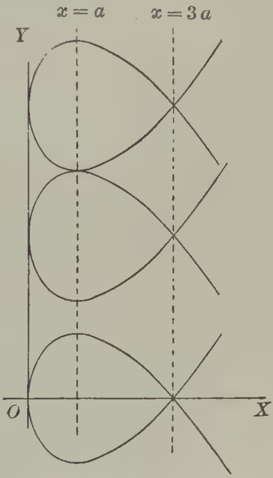

FIG. 3.

38. Equation of the cuspidal locus.

The general solution $\phi(x, y, c)=0$ may represent a set of curves each of which has a cusp. These curves being supposed drawn, the cusps will lie on a curve called *the cuspidal locus*. It is evident that in the limit two consecutive curves of the system will have their cusps coincident upon the cuspidal locus, the cusps thus being among the ultimate points of intersection; and hence the cuspidal locus will appear in the locus of the c discriminant relation. Moreover, the p's at the cusps of consecutive curves will evidently be equal; and therefore the cuspidal locus will appear in the locus of the p discriminant relation. Like the nodal locus, it will not, in general, be the envelope.

Ex. 1. The differential equation

$$p^2 + 2xp - y = 0 \tag{1}$$

has for its general solution

$$(2x^3 + 3xy + c)^2 - 4(x^2 + y)^3 = 0. \tag{2}$$

The p discriminant relation is

$$x^2 + y = 0, \tag{3}$$

and the c discriminant relation is

$$(x^2 + y)^3 = 0.$$

Equation (1) is not satisfied by (3), and hence there is no singular solution; $x^2 + y = 0$ is a cusp locus.

Ex. 2. The equation $8ap^3 = 27y$

has for its general solution $ay^2 = (x - c)^3$;

the p discriminant relation is $y = 0$,

and the c discriminant relation is $y^4 = 0$.

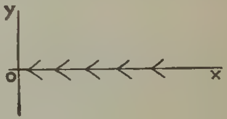

FIG. 4.

The equation $y = 0$ satisfies the differential equation, and therefore is the singular solution. It is also the equation of the cusp locus. Figure 4 illustrates this example. This is one of the very exceptional cases where the cusp locus coincides with the envelope.

39. Summary. When the loci discussed above exist, then in the p discriminant relation will appear the equations of the envelope locus, of the cuspidal locus, and of the tac-locus; and in the c discriminant equation will appear the equations of the envelope locus, of the cuspidal locus, and of the nodal locus.*

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* See Edwards, *Differential Calculus*, Arts. 364–366; Johnson, *Differential Equations*, Arts. 45–54; Forsyth, *Differential Equations*, Arts. 23–30; an article by Cayley, "On the theory of the singular solutions of differential equations of the first order" (*Messenger of Mathematics*, Vol. II. [1872], pp. 6–12); an article by J. W. L. Glaisher, "Examples illustrative of Cayley's theory of singular solutions" (*Messenger of Mathematics*, Vol. XII. [1882], pp. 1–14).

The p discriminant relation contains the equations of the envelope, cuspidal and tac loci, once, once, and twice respectively; and the c discriminant relation contains the equations of the envelope, cuspidal and nodal loci, once, three times, and twice respectively.*

EXAMPLES ON CHAPTER IV.

Solve and find the singular solutions of the following equations:

1. $xp^2 - 2yp + ax = 0$.

2. $x^3p^2 + x^2yp + a^3 = 0$.

3. $y^2 - 2pxy + p^2(x^2 - 1) = m^2$.

4. $y = xp + \sqrt{b^2 + a^2p^2}$.

5. $y = xp - p^2$.

6. Examine Exs. 2, 4, 20, 26, Chap. III., for singular solutions.

7. Solve $4p^2 = 9x$, and examine for singular solution.

8. Investigate for singular solution

$$4x(x-1)(x-2)p^2 - (3x^2 - 6x + 2)^2 = 0.$$

9. Solve and examine for singular solution $(8p^3 - 27)x = 12p^2y$.

10. $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$.

11. $(px - y)(x - py) = 2p$.

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* This is proved in an article by M. J. M. Hill, "On the c and p discriminant of ordinary integrable differential equations of the first order" (*Proc. Lond. Math. Soc.*, Vol. XIX. [1888], pp. 561–589). This article supplements Cayley's, mentioned above.

Professor Chrystal has shown that the p discriminant locus is in general a cuspidal locus for the family of integral curves. (*Nature*, Vol. LIV., 1896, p. 191.)

CHAPTER V.

APPLICATIONS TO GEOMETRY, MECHANICS, AND PHYSICS.

40. The student will remember that, after deducing the methods of solving various kinds of algebraic equations and working through lists of these equations, he made practical applications of the knowledge and skill thus acquired, in the solution of problems. In the process of finding the solution of one of these problems, there were three steps: first, forming the equations that expressed the relations existing between the quantities considered in the problem; second, solving these equations; and third, interpreting the algebraic solution.

In the case of differential equations, the same procedure will be followed. The three preceding chapters have shown methods of solving differential equations of the first order. This chapter will be concerned with practical problems, the solution of which will require the use of these methods. The problems will be chosen for the most part from geometry and mechanics; and it is presupposed that the student possesses as much knowledge of these subjects as can be acquired from elementary text-books on the differential calculus and mechanics.

As in the case of algebraic problems, there are three steps in obtaining the solution of the problems now to be considered:

First, forming the differential equations that express the relations existing between the variables involved.

Second, finding the solution of these equations.

Third, interpreting this solution.

There will be only two variables involved in each of these problems, and hence but a single equation will be required. The choice of examples for this chapter is restricted, because differential equations of the first order only have so far been treated.

41. Geometrical problems. The student should review the articles in the differential calculus that deal with curves; in particular, those articles that treat of the tangent and normal, their directions, lengths, and projections, and the articles that discuss curvature and the radius of curvature. This review will be of great service in helping him to express the data of the problem in the form of an equation, and to interpret the solution of this equation. The character of the geometrical problems and the method of their solution will in general be as follows. A curve will be described by some property belonging to it, and from this its equation will have to be deduced. This is like what is done in analytic geometry, but here the statement of the property will take the form of a differential equation; the solution of this differential equation will be the required equation of the curve.

42. Geometrical data. The following list of some of the principal geometrical deductions of the differential calculus is given for reference. It will be of service in forming the differential equations which express the conditions stated in the problems, or, in other words, give the properties belonging to the curves whose equations are required.

Suppose that the equation of a curve, rectangular co-ordinates being chosen, is

$$y = f(x), \text{ or } F(x, y) = 0,$$

and that (x, y) is any point on this curve. Then $\frac{dy}{dx}$ is the slope of the tangent at the point (x, y), *i.e.* the tangent of the angle that the tangent line there makes with the x-axis; $-\frac{dx}{dy}$ is the slope of the normal; the equation of the tangent at (x, y),

X, Y, being the current co-ordinates, is $Y - y = \frac{dy}{dx}(X - x)$; and the equation of the normal is $Y - y = -\frac{dx}{dy}(X - x)$; the intercept of the tangent on the axis of x is $x - y\frac{dx}{dy}$; the intercept of the tangent on the axis of y is $y - x\frac{dy}{dx}$; the length of the tangent, that is, the part of the tangent between the point and the x-axis, is $y\sqrt{1 + \left(\frac{dx}{dy}\right)^2}$; the length of the normal is $y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$; the length of the subtangent is $y\frac{dx}{dy}$; the length of the subnormal is $y\frac{dy}{dx}$; the differential of the length of the arc is $\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$, or $\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$; the differential of the area is $y\,dx$ or $x\,dy$.

Again, let the equation of the curve in polar co-ordinates be

$$f(r, \theta) = 0, \text{ or } r = F(\theta),$$

and (r, θ) be any point on the curve. Then the tangent of the angle between the radius vector and the part of the tangent to the curve at (r, θ) drawn back towards the initial line, is $r\frac{d\theta}{dr}$; if θ is the vectorial angle, ψ the angle between the radius vector and the tangent at (r, θ), and ϕ the angle that this tangent makes with the initial line, $\phi = \psi + \theta$; the length of the polar subtangent is $r^2\frac{d\theta}{dr}$; the length of the polar subnormal is $\frac{dr}{d\theta}$; the differential of the length of the arc is $\sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\,dr$, or $\sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\,d\theta$; if p denote the length of the perpendicular from the pole upon the tangent,* then

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* Williamson, *Differential Calculus*, Art. 183; Edwards, *Differential Calculus for Beginners*, Art. 95.

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2;$$

that is,
$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2,\text{ where } u=\frac{1}{r}.$$

43. Examples.

Ex. 1. Determine the curve whose subtangent is n times the abscissa of the point of contact; and find the particular curve which passes through the point (2, 3).

Let (x, y) be any point upon the curve. The subtangent is $y\,\frac{dx}{dy}$. Therefore, the condition that must be satisfied at any point of the required curve, in other words, the given property of the curve, is expressed by the equation

$$y\,\frac{dx}{dy}=nx.$$

Integration gives $n \log y = \log cx$, whence,

$$y^n = cx.$$

This represents a family of curves, each of which passes through the origin. For the particular curve that passes through (2, 3), c must be $\frac{3^n}{2}$, and the equation is

$$2\,y^n = 3^n\,x.$$

When $n = 1$, the required curve is any one of the straight lines which pass through the origin; the equation of the particular line through (2, 3) is

$$2\,y = 3\,x.$$

When $n = 2$, the curves having the given property are the parabolas whose vertices are at the origin, and whose axes coincide with the x-axis; the particular parabola through (2, 3) has the equation

$$2\,y^2 = 9\,x.$$

When $n = \frac{3}{2}$, the required curve is any one of the system of semicubical parabolas that have their vertices at the origin and their axes coinciding with the axis of y.

What curves have the given property when $n = \frac{1}{2}$? When $n = \frac{2}{3}$?

Ex. 2. Find the curve in which the perpendicular upon the tangent from the foot of the ordinate of the point of contact is constant and equal to a; and determine the constant of integration in such a manner that the curve shall cut the axis of y at right angles.

Let (x, y) be any point on the curve. The equation of the tangent at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x);$$

the length of the perpendicular from $(x, 0)$, the foot of the ordinate, upon the tangent is $\dfrac{-y}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}$.

Therefore, the given property of the curve is expressed by the equation

$$(1)\quad \frac{-y}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = a; \quad \text{from this,} \quad (2)\quad \frac{a\,dy}{\sqrt{y^2-a^2}} = dx;$$

integration gives * $\qquad \cosh^{-1}\dfrac{y}{a} = \dfrac{x}{a} + c;$

whence $\qquad (3)\quad \dfrac{y}{a} = \cosh\left(\dfrac{x}{a} + c\right).$

It is also required that there be found the particular one of these curves that cuts the y-axis at right angles. This means that for this curve, $\dfrac{dy}{dx} = 0$ when $x = 0$. Now differentiation of (3) gives

$$\frac{1}{a}\frac{dy}{dx} = \frac{1}{a}\sinh\left(\frac{x}{a} + c\right);$$

therefore $c = 0$; and hence

$$\frac{y}{a} = \cosh\frac{x}{a},$$

the equation of the catenary.

Ex. **3.** Determine the curve in which the subtangent is n times the subnormal.

Ex. **4.** Determine the curve in which the length of the arc measured from a fixed point A to any point P is proportional to the square root of the abscissa of P.

Ex. **5.** Find the curve in which the polar subnormal is proportional to the sine of the vectorial angle.

Ex. **6.** Find the curve in which the polar subtangent is proportional to the length of the radius vector.

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* See McMahon, *Hyperbolic Functions* (Merriman and Woodward, *Higher Mathematics*, Chap. IV.), Arts. 14, 15, 26, 39; Edwards, *Integral Calculus for Beginners*, Arts 28–44.

44. Problems relating to trajectories. An important group of geometrical problems is that which deals with trajectories. A *trajectory* of a family of curves is a line that cuts all the members of the family according to a given law; for example, the line which cuts all the curves of the family at points equidistant from the x-axis, the distance being measured along the curves of the family. Another example of a trajectory is the line that cuts the curves of the family at a constant angle. When the angle is a right angle, the trajectories are called orthogonal trajectories; when it is other than a right angle, the trajectories are said to be oblique. Only these two classes of trajectories will here be discussed.

45. Trajectories, rectangular co-ordinates. Suppose that

$$f(x, y, a) = 0 \tag{1}$$

is the equation of the given system of curves, a being the arbitrary parameter; and that α is the angle at which the trajectories are to cut the given curves. The elimination of a from (1) gives an equation of the form

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0, \tag{2}$$

the differential equation of the family of curves.

Now through any point (x, y) there pass a curve of the given system and one of the trajectories, cutting each other at an angle α. If m is the slope of the tangent to the trajectory at this point, then

$$m = \frac{\dfrac{dy}{dx} - \tan\alpha}{1 + \dfrac{dy}{dx}\tan\alpha}. \tag{3}$$

By definition m is $\frac{dy}{dx}$ for the trajectory; hence the differential equation of the system of trajectories is obtained by substituting this value of m for $\frac{dy}{dx}$ in (2); this gives

$$\phi\left[x,\ y,\ \frac{\dfrac{dy}{dx}-\tan\alpha}{1+\dfrac{dy}{dx}\tan\alpha}\right]=0, \tag{4}$$

for the differential equation of the system of trajectories; and the solution of this is the integral equation.

If α is a right angle,

$$m=-\frac{dx}{dy};$$

and hence the differential equation of the system of *orthogonal* trajectories is obtained by substituting $-\dfrac{dx}{dy}$ for $\dfrac{dy}{dx}$ in (2); this gives

$$\phi\left(x,\ y,\ -\frac{dx}{dy}\right)=0. \tag{5}$$

Integration will give the equation in the ordinary form.

46. Orthogonal trajectories, polar co-ordinates. Suppose that

$$f(r,\ \theta,\ c)=0 \tag{1}$$

is the polar equation of the given curve, and that

$$\phi\left(r,\ \theta,\ \frac{dr}{d\theta}\right)=0 \tag{2}$$

is the corresponding differential equation, obtained by eliminating the arbitrary constant c. The tangent of the angle between the radius vector and the tangent to a curve of the given system at any point $(r,\ \theta)$ is $r\dfrac{d\theta}{dr}$. If m is the tangent of the angle between this radius vector and the tangent to the trajectory through that point,

$$m=-\frac{1}{r}\frac{dr}{d\theta},$$

since the tangents of the curve and its trajectory are at right angles to each other. Hence the differential equation of the required trajectory is obtained by substituting $-\frac{1}{r}\frac{dr}{d\theta}$ for $r\frac{d\theta}{dr}$, or, what comes to the same thing, $-r^2\frac{d\theta}{dr}$ for $\frac{dr}{d\theta}$ in (2); this gives

$$\phi\left(r,\,\theta,\,-r^2\frac{d\theta}{dr}\right)=0 \tag{3}$$

as the differential equation of the required system of trajectories.

47. Examples.

Ex. 1. Find the equation of the curve which cuts at a constant angle whose tangent is $\frac{m}{n}$ all the circles touching a given straight line at a given point.

Take the given point for the origin, the given line for the y-axis, and the perpendicular to it at the point for the x-axis. The given system of circles then consists of the circles which pass through the origin and have their centres on the x-axis; its equation is

$$y^2+x^2-2\,ax=0, \tag{1}$$

a being the variable parameter. The elimination of a gives the differential equation of the system of circles; namely,

$$\frac{dy}{dx}=\frac{y^2-x^2}{2\,xy}. \tag{2}$$

The differential equation of the system of trajectories is obtained by substituting for $\frac{dy}{dx}$ in equation (2) the expression

$$\frac{\dfrac{dy}{dx}+\dfrac{m}{n}}{1-\dfrac{m}{n}\dfrac{dy}{dx}};$$

and this gives on reduction

$$(nx^2-ny^2+2\,mxy)dx+(my^2-mx^2+2\,nxy)dy=0. \tag{3}$$

The integration of this homogeneous equation gives

$$x^2+y^2=2\,c(my+nx), \tag{4}$$

c being the constant of integration; this represents another system of circles.

The trajectory is orthogonal if $n=0$; equation (4) then becomes

$$x^2+y^2=c_1y,$$

which represents the orthogonal system of circles; these circles pass through the origin and have their centres on the y-axis.

Ex. 2. Find the orthogonal trajectories of the system of curves

$$r^n \sin n\theta = a^n.$$

Differentiation eliminates the parameter a, and gives

$$\frac{dr}{d\theta} + r \cot n\theta = 0,$$

the differential equation of the system.

The differential equation of the system of trajectories is obtained by substituting $-r^2\frac{d\theta}{dr}$ for $\frac{dr}{d\theta}$; this gives

$$-r^2\frac{d\theta}{dr} + r \cot n\theta = 0;$$

separating the variables, integrating, and simplifying,

$$r^n \cos n\theta = c,$$

c being an arbitrary constant; this is the equation of the system of orthogonal curves.

Ex. 3. Find the orthogonal trajectories of a series of parabolas whose equation is $y^2 = 4\,ax$.

Ex. 4. Find the orthogonal trajectories of the series of hypocycloids $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Ex. 5. Find the equation of the system of orthogonal trajectories of a series of confocal and coaxial parabolas $r = \frac{2\,a}{1 + \cos\theta}$.

Ex. 6. Find the orthogonal trajectories of the series of curves.

$$r = a + \sin 5\,\theta.$$

Ex. 7. Given the set of lines $y = cx$, c being arbitrary, find all the curves that cut these lines at a constant angle θ.

48. Mechanical and physical problems. The student should read in some text-book on mechanics the articles in which the elementary principles and formulæ relating to force and motion are enunciated and deduced. The truth of the following definitions and formulæ will be apparent to one who understands the first principles of the calculus and the principles of mechanics as set forth in elementary works that do not employ the calculus.

If s denotes the length of the path described by a particle moving in a straight line for any period of time;

t, the time of motion, usually estimated in seconds; and

v, the velocity of the moving particle at any particular point or instant; then will

$\dfrac{ds}{dt} = v$, and

$\dfrac{dv}{dt} =$ the acceleration of the moving particle at any point of its path.

Ex. 1. A body falls from rest; assuming that the resistance of the air is proportional to the square of the velocity, find

(*a*) its velocity at any instant;

(*b*) the distance through which it has fallen.

In this case the equation for the acceleration is

$$\frac{dv}{dt} = g - \kappa v^2, \text{ or, putting } \frac{n^2}{g} \text{ for } \kappa,$$

$$g\frac{dv}{dt} = g^2 - n^2v^2,$$

whence $$\frac{gdv}{g^2 - n^2v^2} = dt.$$

Integrating, $\tanh^{-1}\dfrac{nv}{g} = nt + c$; whence, $\dfrac{nv}{g} = \tanh(nt + c)$.

But $c = 0$, since $v = 0$ when $t = 0$.

Therefore $$v = \frac{ds}{dt} = \frac{g}{n}\tanh nt;$$

whence, on integration,

$$s + c = \frac{g}{n^2}\log\cosh nt.$$

But $s = 0$ when $t = 0$, $\therefore\ c = 0$;

therefore $$s = \frac{g}{n^2}\log\cosh nt.$$

Ex. 2. Find the distance passed over in time t by a particle whose acceleration is constant, determining the constants of integration so that at the time $t = 0$, v_0 is the velocity and s_0 the distance of the particle from the point from which distance is measured.

Ex. 3. The velocity possessed by a body after falling vertically from rest through a distance s is found to be $\sqrt{2gs}$. Find the height through which it has fallen in terms of the time.

EXAMPLES ON CHAPTER V.

1. Determine the curve in which the length of the subnormal is proportional to the square of the ordinate.

2. Determine the curve in which the part of the tangent intercepted by the axes is a constant a. [Hint: Find the singular solution.]

3. Determine the curve in which the length of the subnormal is proportional to the square of the abscissa.

4. Find the equation of the curve for which a differential of the arc is κ times the differential of the angle made by its tangent with the x-axis, multiplied by the cosine of this angle; and determine the constant of integration so that the curve touches the x-axis at the point from which the arc is measured.

5. Find the equation of the curve where the length of the perpendicular from the pole upon the tangent is constant and equal to $\frac{1}{\kappa}$.

6. Find the equation of the system of curves that make an angle whose tangent is $\frac{m}{n}$ with the series of parallel lines $x \cos \alpha + y \sin \alpha = p$, p being the variable parameter.

7. Find the orthogonal trajectories of the system of parabolas $y = ax^2$.

8. Find the orthogonal trajectories of the system of circles touching a given straight line at a given point.

9. Find the orthogonal trajectories of $\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1$, where λ is arbitrary.

10. Find the orthogonal trajectories of the series of hyperbolas $xy = \kappa^2$.

11. Determine the orthogonal trajectories of the system of curves $r^n = a^n \cos n\theta$; therefrom find the orthogonal trajectories of the series of lemniscata $r^2 = a^2 \cos 2\theta$.

12. Find the orthogonal trajectories of $\left(r + \frac{\kappa^2}{r}\right) \cos \theta = \alpha$, α being the parameter.

13. Find the orthogonal trajectories of the series of logarithmic spirals $r = a^\theta$, where a varies.

14. Determine the curve whose tangent cuts off from the co-ordinate axes intercepts whose sum is constant.

15. The perpendiculars from the origin upon the tangents of a curve are of constant length a. Find the equation of the curve.

16. Find the equation of the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa of the point of contact.

17. Find the equation of a curve such that the projection of its ordinate upon the normal is equal to the abscissa.

18. Find the equation of the curve in which, if any point P be taken, the perpendicular let fall from the foot of its ordinate upon its radius vector shall cut the y-axis where the latter is cut by the tangent to the curve at P.

19. Find the curve in which the angle between the radius vector and the tangent is n times the vectorial angle. What is the curve when $n = 1$? When $n = \frac{1}{2}$?

20. Determine the curve in which the normal makes equal angles with the radius vector and the initial line.

21. Find the curve the length of whose arc measured from a given point is a mean proportional between the ordinate and twice the abscissa.

22. Find the equation of the curve in which the perpendicular from the pole upon the tangent at any point is k times the radius vector of the point.

23. If $\frac{1}{p^2} = \frac{1}{a^2(1-e^2)}\left(\frac{2a}{r} - 1\right)$, find the equation of the curve, r being the radius vector of any point of the curve, and p the perpendicular from the pole upon the tangent at that point.

24. Find the orthogonal trajectories of the cardioids $r = a(1 - \cos\theta)$.

25. Show that the system of confocal and coaxial parabolas $y^2 = 4a(x + a)$ is self-orthogonal.

26. Show that a system of confocal conics is self-orthogonal.

27. Find the curve such that the rectangle under the perpendiculars from two fixed points on the normals be constant.

28. Find the curve in which the product of the perpendiculars drawn from two fixed points to any tangent is constant.

29. The product of two ordinates drawn from two fixed points on the x-axis to the tangent of a curve is constant and equal to κ^2. Find the equation of the curve.

30. Determine the curve in which the area enclosed between the tangent and the co-ordinate axes is equal to a^2.

31. Find a curve such that the area included between a tangent, the x-axis, and two perpendiculars upon the tangent from two fixed points on the x-axis is constant and equal to κ^2.

32. The parabola $y^2 = 4\,ax$ rolls upon a straight line. Determine the curve traced by the focus.

33. Determine the curve in which $s = ax^2$.

34. The equation of electromotive forces for an electric circuit containing resistance and self-induction is

$$E = Ri + L\frac{di}{dt},$$

where E is the electromotive force given to the circuit, R the resistance, and L the coefficient of induction. Find the current i: (a) when $E = f(t)$; (b) when $E = 0$; (c) when $E =$ a constant; (d) when E is a simple harmonic function of the time, $E_m \sin \omega t$, where E_m is the maximum value of the impressed electromotive force, and ω is $2\,\pi$ times the frequency of alternation; (e) when $E = E_1 \sin \omega t + E_2 \sin (b\omega t + \theta)$.

35. The equation of electromotive forces in terms of the current i, for an electric circuit having a resistance R, and having in series with that resistance a condenser of capacity C, is $E = Ri + \int \frac{idt}{C}$, which reduces on differentiation to the form

$$\frac{di}{dt} + \frac{i}{RC} = \frac{1}{R}\frac{dE}{dt},$$

E being the electromotive force. Find the current i: (a) when $E = f(t)$; (b) when $E = 0$; (c) when $E =$ a constant; (d) when $E = E_m \sin \omega t$.

36. Given that the equation of electromotive forces in the circuit of the last example, in terms of the charge q, is

$$E = R\frac{dq}{dt} + \frac{q}{C},$$

find q: (a) when $E = f(t)$; (b) when $E = 0$; (c) when $E =$ a constant; (d) when $E = E_m \sin \omega t$.

37. The acceleration of a moving particle being proportional to the cube of the velocity and negative, find the distance passed over in time t, the initial velocity being v_0, and the distance being measured from the position of the particle at the time $t = 0$.

CHAPTER VI.

LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS.

49. Linear equations defined. The complementary function, the particular integral, the complete integral. Equations of an order higher than the first have now to be considered. This chapter and the next will deal with a single class of these equations; namely, linear differential equations. In these, the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients all being constants or functions of x. The general form of the equation is

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\cdots+P_ny=X, \tag{1}$$

where X and the coefficients $P_1, P_2, \cdots, P_n$, are constants or functions of x. If the derivative of highest order, $\frac{d^ny}{dx^n}$, has a coefficient other than unity, the members of the equation can be divided by this coefficient, and then the equation will be in the form (1). The linear equation of the first order has been treated in Art. 20.

It will first be shown that the complete solution of (1) contains, as part of itself, the complete solution of

$$\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_ny=0. \tag{2}$$

If $y=y_1$ be an integral of (2), then, as will be seen on substitution in (2), $y=c_1y_1$, c_1 being an arbitrary constant, is also an integral; similarly if $y=y_2$, $y=y_3$, $\cdots$, $y=y_n$, be integrals of (2), then $y=c_2y_2$, $\cdots$, $y=c_ny_n$, where $c_2, \cdots, c_n$ are arbitrary constants, are all integrals. Moreover, substitution will show that

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \tag{3}$$

is an integral. If $y_1, y_2, \cdots, y_n$, are linearly independent,* (3) is the complete integral of (2), since it contains n arbitrary constants and (2) is of order n.

If $y = u$ be a solution of (1), then

$$y = Y + u, \tag{4}$$

where $$Y = c_1y_1 + c_2y_2 + \cdots + c_ny_n,$$

is also a solution of (1); for the substitution of Y for y in the first member of (1) gives zero, and that of u for y, by hypothesis, gives X. As the solution (4) contains n arbitrary constants, it is the complete solution of equation (1). The part Y is called *the complementary function;* and the part u is called *the particular integral.*† The general or *complete* solution is the sum of the complementary function and the particular integral.

50. The linear equation with constant coefficients and second member zero. The equation

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_ny = 0, \tag{1}$$

where the coefficients $P_1, P_2, \cdots, P_n$, are constants, will first be treated.‡

On the substitution of e^{mx} for y, the first member of this equation becomes $(m^n + P_1m^{n-1} + \cdots + P_n)e^{mx}$; and this will be equal to zero if

$$m^n + P_1m^{n-1} + \cdots + P_n = 0. \tag{2}$$

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* See Note F for the criterion of the linear independence of the integrals $y_1, y_2, \cdots, y_n$.

† This use of the term *particular integral* is to be distinguished from that indicated in Art. 4.

‡ The method of solving the linear differential equation with constant coefficients, shown in this article, is due to Leonhard Euler (1707–1783), one of the most distinguished mathematicians of the eighteenth century.

This may be called *the auxiliary equation.* Therefore, if m have a value, say m_1, that satisfies (2), $y = e^{m_1x}$ is an integral of (1); and if the n roots of (2) be $m_1, m_2, \cdots m_n$, the complete solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}. \tag{3}$$

Ex. 1. Solve $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - 54y = 0$.

Here equation (2) is $\quad m^2 + 3m - 54 = 0;$

solving for m, $\quad m = 6, \ -9.$

Hence the general solution of the equation is $y = c_1e^{6x} + c_2e^{-9x}$.

Ex. 2. If $\dfrac{d^2y}{dx^2} - m^2y = 0$, show that

$$y = c_1e^{mx} + c_2e^{-mx} = A\cosh mx + B\sinh mx.\text{*}$$

Ex. 3. Solve $2\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} - 12x = 0$.

Ex. 4. Solve $9\dfrac{d^2x}{dz^2} + 18\dfrac{dx}{dz} - 16x = 0$.

51. Case of the auxiliary equation having equal roots. When two roots of (2) Art. 50 are *equal*, say m_1 and m_2, solution (3) becomes

$$y = (c_1 + c_2)e^{m_1x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}.$$

But, since $c_1 + c_2$ is equivalent to only a single constant, this solution will have $(n - 1)$ arbitrary constants; and hence is not the general solution. In order to obtain the complete solution in this case, suppose that

$$m_2 = m_1 + h;$$

the terms of the solution corresponding to m_1, m_2, will then be

$$y = c_1e^{m_1x} + c_2e^{(m_1+h)x},$$

which can be written $y = e^{m_1x}(c_1 + c_2e^{hx})$.

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* See McMahon, *Hyperbolic Functions* (Merriman and Woodward, *Higher Mathematics*, Chap. IV.), Arts. 14 (Prob. 30), 17, 39.

On expanding e^{hx} by the exponential series, this becomes

$$
\begin{aligned}
y &= e^{m_1x}\left[c_1 + c_2\left(1 + hx + \frac{h^2x^2}{1\cdot 2} + \cdots\right)\right] \\
&= e^{m_1x}\left[c_1 + c_2 + c_2hx\left(1 + \frac{hx}{1\cdot 2} + \frac{h^2x^2}{1\cdot 2\cdot 3} + \cdots\right)\right] \\
&= e^{m_1x}\left(A + Bx + \frac{c_2h^2x^2}{1\cdot 2} + \text{terms proceeding in ascending powers of } h\right),
\end{aligned}
$$

where $A = c_1 + c_2$, and $B = c_2h$.

Now let h approach 0, and solution (3) Art. 50 takes the form

$$y = e^{m_1x}(A + Bx) + c_3e^{m_3x} + \cdots + c_ne^{m_nx}.$$

As h approaches zero, c_1 and c_2 can be taken in such a way that A and B will be finite.

If the auxiliary equation have three roots equal to m_1, by similar reasoning it can be shown that the corresponding solution is

$$y = e^{m_1x}(c_1 + c_2x + c_3x^2);$$

and, if it have r equal roots, that the corresponding solution is

$$y = e^{m_1x}(c_1 + c_2x + \cdots + c_rx^{r-1}).\text{*}$$

The form of the solution in the case of repeated roots of the auxiliary equation is deduced in another way in Art. 55.

Ex. 1. Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0.$

Ex. 2. Solve $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0.$

52. Case of the auxiliary equation having imaginary roots. When equation (2) Art. 50 has a pair of *imaginary roots*, say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ (i being used to denote $\sqrt{-1}$), the corresponding part of the solution can be put in a real form simpler, and hence more useful, than the exponential form of Art. 50.

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[*] See George Boole, *Differential Equations*, Chap. IX., Art. 7. The separate integrals, e^{m_1x}, xe^{m_1x}, $x^2e^{m_1x}$, $\cdots$, are analogous to the equal roots of an algebraic equation.

For, $$c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}=e^{\alpha x}(c_1e^{i\beta x}+c_2e^{-i\beta x})$$
$$=e^{\alpha x}\{c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)\}$$
$$=e^{\alpha x}(A\cos\beta x+B\sin\beta x)$$
$$=(\cosh\alpha x+\sinh\alpha x)(A\cos\beta x+B\sin\beta x).$$

If a pair of imaginary roots occurs twice, the corresponding solution is $$y=(c_1+c_2x)\,e^{(\alpha+i\beta)x}+(c_3+c_4x)\,e^{(\alpha-i\beta)x},$$ which reduces to $y=e^{\alpha x}\{(A+A_1x)\cos\beta x+(B+B_1x)\sin\beta x\}$.

Ex. **1.** Solve $\dfrac{d^2y}{dx^2}+8\dfrac{dy}{dx}+25y=0.$

The auxiliary equation is $m^2+8m+25=0$, the roots of which are $m=-4\pm3i$; and the solution is $y=e^{-4x}(c_1\cos3x+c_2\sin3x)$.

Ex. **2.** If $\dfrac{d^4y}{dx^4}-m^4y=0$, show that

$$y=c_1\cos mx+c_2\sin mx+c_3\cosh mx+c_4\sinh mx.$$

Ex. **3.** Solve $\dfrac{d^4y}{dx^4}-4\dfrac{d^3y}{dx^3}+8\dfrac{d^2y}{dx^2}-8\dfrac{dy}{dx}+4y=0.$

53. The symbol D. By using the symbol D for the differential operator $\dfrac{d}{dx}$, equation (1) Art. 50 can be written *

$$(D^n+P_1D^{n-1}+\cdots+P_n)\,y=0, \tag{1}$$

or, briefly, $$f(D)\,y=0. \tag{2}$$

The symbolic coefficient of y in (1) is the same function of D that the first member of equation (2) Art. 50 is of m; and, therefore, the roots of the latter equation being $m_1, m_2, \cdots, m_n$, equation (1) may be written

$$(D-m_1)(D-m_2)\cdots(D-m_n)\,y=0. \tag{3}$$

Hence the integral of (1) can be found by putting its symbolic coefficient equal to zero, and solving for D as if it were an ordinary algebraic quantity, without any regard to its use as an operator; and then proceeding as in Art. 50 after the roots of equation (2) of that article had been found. Moreover, it is thus apparent that the complete solution of (1) or (3) is made up of the solutions of

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* See note K, page 208, for remarks on the symbol D.

$$(D-m_1)y=0,\ (D-m_2)y=0,\ \cdots,\ (D-m_n)y=0.$$

This symbol D will be of great service.

54. Theorem concerning D. One of the theorems relating to D is, that when the coefficient of y in (1) Art. 53 is factored as if D were an ordinary algebraic quantity, then the original differential equation will be obtained when D is given its operative character, no matter in what order the factors are taken. Thus, an equation of the second order

$$\frac{d^2y}{dx^2}-(\alpha+\beta)\frac{dy}{dx}+\alpha\beta y=0,$$

when expressed in the symbolic form, is

$$\{D^2-(\alpha+\beta)D+\alpha\beta\}y=0;$$

this on factoring becomes $(D-\alpha)(D-\beta)y=0.$

Replacing D by $\frac{d}{dx}$, the latter equation becomes

$$\left(\frac{d}{dx}-\alpha\right)\left(\frac{d}{dx}-\beta\right)y=0.$$

Operating on y with $\frac{d}{dx}-\beta$, this becomes

$$\left(\frac{d}{dx}-\alpha\right)\left(\frac{dy}{dx}-\beta y\right)=0;$$

and, operating on the second factor with the first,

$$\frac{d^2y}{dx^2}-(\alpha+\beta)\frac{dy}{dx}+\alpha\beta y=0.$$

If the factors had been written in the reverse order, $(D-\beta)(D-\alpha)y=0$, and expanded as above, the same result would have been obtained. It is easily shown that the theorem holds for an equation of the third and any higher order. It will be noted that the symbolic factors, when used as operators, are taken in order from right to left. Other theorems relating to D will be proven when a reference to them happens to be required.*

55. Another way of finding the solution when the auxiliary equation has repeated roots. The form of the solution when the auxiliary equation (2) Art. 50 has repeated roots can be found in another way; namely, by employing the symbol D. According to Art. 53, the solutions corresponding to the two equal roots m_1 of this equation are the solutions of

$$(D - m_1)^2 y = 0.$$

On writing this in the form $(D - m_1)\{(D - m_1)y\} = 0$ and putting v for $(D - m_1)y$, the above equation becomes

$$(D - m_1)v = 0,$$

the solution of which is $v = c_1 e^{m_1 x}$. Replacing v by its value $(D - m_1)y$,

$$(D - m_1)y = c_1 e^{m_1 x},$$

which is the linear equation of the first order considered in Art. 20; its solution is $y = e^{m_1 x}(c_2 + c_1 x)$.

Similarly the solutions corresponding to three equal roots m_1 are the solutions of

$$(D - m_1)^3 y = 0,$$

which may be written

$$(D - m_1)(D - m_1)^2 y = 0.$$

On putting v for $(D - m_1)^2 y$, solving for v, and replacing the value of v as before there is obtained

$$(D - m_1)^2 y = c_1 e^{m_1 x}.$$

Putting ω for $(D - m_1)y$ and proceeding as before,

$$(D - m_1)y = e^{m_1 x}(c_1 x + c_2),$$

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* See Forsyth, *Differential Equations*, Arts. 31–35, for fuller information concerning the properties of D.

the solution of which is

$$y = e^{m_1x}(cx^2 + c_2x + c_3), \text{ where } c = \frac{c_1}{2}.$$

It is obvious that if m_1 is repeated r times, the corresponding integrals are

$$y = e^{m_1x}(c_1 + c_2x + \cdots + c_rx^{r-1}).$$

56. The linear equation with constant coefficients and second member a function of x. In this article will be considered the equation

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_ny = X, \tag{1}$$

the first member of which is the same as that of equation (1) Art. 50, and the second member a function of x. It was pointed out in Art. 49 that the complete integral of (1) consists of two parts, — a complementary function and a particular integral, the complementary function being the complete solution of the equation formed by putting the first member of (1) equal to zero. The problem now is to devise a method for obtaining the particular integral.

In the symbolic notation, (1) becomes

$$f(D)y = X \tag{2}$$

and the particular integral is written $y = \frac{1}{f(D)}X$.

57. The symbolic function $\frac{1}{f(D)}$. It is necessary to define $\frac{1}{f(D)}X$, which, as yet, is a mere symbol without meaning. For this purpose it may be said: $\frac{1}{f(D)}X$ is that function of x which, when operated upon by $f(D)$, gives X. The operator $\frac{1}{f(D)}$, according to this definition, is the inverse of the operator $f(D)$. It can be shown from this definition and Art. 54, that $\frac{1}{f(D)}$ can be broken up into factors which may be taken in any order, or into partial fractions.

For example, the particular integral of the equation

$$\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = X$$

is
$$\frac{1}{D^2 - (\alpha + \beta) D + \alpha\beta} X;$$

and this can be put in the form

$$\frac{1}{(D - \alpha)(D - \beta)} X.$$

Now apply $(D - \alpha)(D - \beta)$ to this, arranging the factors of the latter operator conveniently, as is allowable by Art. 54; this gives

$$(D - \beta)(D - \alpha)\frac{1}{(D - \alpha)(D - \beta)} X;$$

and since $D - \alpha$, acting upon $\frac{1}{D - \alpha} \cdot \frac{1}{D - \beta} X$, must by definition give $\frac{1}{D - \beta} X$, this becomes $D - \beta \cdot \frac{1}{D - \beta} X$, which is X by the definition of $\frac{1}{f(D)}$. This reduction shows that the particular integral might equally well have been written

$$\frac{1}{(D - \beta)(D - \alpha)} X.$$

Also, $\frac{1}{D^2 - (\alpha + \beta)D + \alpha\beta} X$ may be written in the form

$$\frac{1}{\alpha - \beta}\left(\frac{1}{D - \alpha} - \frac{1}{D - \beta}\right) X,$$

which is obtained by resolving the operator into partial fractions. The result of operating upon this with $D^2 - (\alpha + \beta) D + \alpha\beta$ is

$$\frac{1}{\alpha - \beta}\left\{(D - \beta)(D - \alpha)\frac{1}{D - \alpha} X - (D - \alpha)(D - \beta)\frac{1}{(D - \beta)} X\right\},$$

or
$$\frac{1}{\alpha - \beta}\{(D - \beta)X - (D - \alpha)X\};$$

and finally, X.

The statement immediately preceding this example can easily be verified for the general case by a method similar to that used in this particular instance.

58. Methods of finding the particular integral. It is thus apparent that the particular integral of equation (2) Art. 56, namely, $\frac{1}{f(D)}X$, may be obtained in the two following ways:

(*a*) The operator $\frac{1}{f(D)}$ may be factored; then the particular integral will be

$$\frac{1}{D-m_1}\cdot\frac{1}{D-m_2}\cdots\frac{1}{D-m_n}X.$$

On operating with the first symbolic factor, beginning at the right, there is obtained

$$\frac{1}{D-m_1}\cdot\frac{1}{D-m_2}\cdots\frac{1}{D-m_{n-1}}e^{m_nx}\int e^{-m_nx}X\,dx;^*$$

then, on operating with the second and remaining factors in succession, taking them from right to left, there is finally obtained the value of the particular integral, namely,

$$e^{m_1x}\int e^{(m_2-m_1)x}\int\cdots\int e^{-m_nx}X(dx)^n.$$

(*b*) The operator $\frac{1}{f(D)}$ may be decomposed into its partial fractions

$$\frac{N_1}{D-m_1}+\frac{N_2}{D-m_2}+\cdots+\frac{N_n}{D-m_n},$$

and then the particular integral will have the form

$$N_1e^{m_1x}\int e^{-m_1x}X\,dx+N_2e^{m_2x}\int e^{-m_2x}X\,dx+\cdots+N_ne^{m_nx}\int e^{-m_nx}X\,dx.$$

Of these two methods, the latter is generally to be preferred.

Since the methods (*a*) and (*b*) consist altogether of operations of the kind effected by $\frac{1}{D-a}$ upon X, the result of the latter operation should be remembered. Now, $\frac{1}{D-a}X$ is the particular integral of the linear equation of the first order,

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* This is made clear in the last paragraph of this article.

$$\frac{dy}{dx} - ay = X,$$

which has been discussed in Art. 20; its value is

$$e^{ax}\int e^{-ax}\, X dx.$$

The term c_1e^{ax} in the solution of this equation is the complementary function.

Ex. 1. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$.

This equation written in symbolic form is

$$(D^2 - 5D + 6)y = e^{4x},$$

or

$$(D-3)(D-2)y = e^{4x};$$

hence the complementary function is $y = c_1e^{3x} + c_2e^{2x}$; and the particular integral is

$$y = \frac{1}{D-3}\cdot\frac{1}{D-2}e^{4x} = \left(\frac{1}{D-3} - \frac{1}{D-2}\right)e^{4x}$$

$$= e^{3x}\int e^{-3x}e^{4x}dx - e^{2x}\int e^{-2x}e^{4x}dx = e^{4x} - \frac{e^{4x}}{2} = \frac{e^{4x}}{2};$$

and hence the general solution is

$$y = c_1e^{3x} + c_2e^{2x} + \tfrac{1}{2}e^{4x}.$$

Ex. 2. Solve $\frac{d^2y}{dx^2} - y = 2 + 5x$.

Ex. 3. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{2x}$.

Ex. 4. Solve $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 12y = X$.

59. Short methods of finding the particular integrals in certain cases. The terms of the particular integral which correspond to terms of certain special forms that may appear in the second member of the equation, can be obtained by methods that are much shorter than the general methods shown in the last article. The special forms occurring in the second member which will be discussed here are:

(*a*) e^{ax}, where a is any constant;
(*b*) x^m, where m is a positive integer;
(*c*) $\sin ax$, $\cos ax$;
(*d*) $e^{ax}V$, where V is any function of x;
(*e*) xV, where V is any function of x.

60. Integral corresponding to a term of form e^{ax} in the second member. The integral corresponding to e^{ax} in the second member of the equation $f(D)y = X$, is $\frac{1}{f(D)}e^{ax}$; this will be shown to be equal to $\frac{1}{f(a)}e^{ax}$.

Successive differentiation gives $D^n e^{ax} = a^n e^{ax}$; the terms appearing in $f(D)$ are terms of the form D^n, n being an integer; therefore

$$f(D)\,e^{ax} = f(a)\,e^{ax}.$$

Operating on both members with $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}f(D)\,e^{ax} = \frac{1}{f(D)}f(a)\,e^{ax};$$

and this, since $\frac{1}{f(D)}$ and $f(D)$ are inverse operators and $f(a)$ is only an algebraic multiplier, reduces to

$$e^{ax} = f(a)\,\frac{1}{f(D)}\,e^{ax};$$

whence

$$\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}\,e^{ax}.$$

The method fails if a is a root of $f(D) = 0$, for then

$$\frac{1}{f(D)}\,e^{ax} = \infty\,e^{ax}.$$

In this case, the procedure is as follows.

Since a is a root of $f(D) = 0$, $(D - a)$ is a factor of $f(D)$ Suppose that $f(D) = (D - a)\,\phi(D)$; then

$$\frac{1}{f(D)}\,e^{ax} = \frac{1}{(D-a)}\,\frac{1}{\phi(D)}\,e^{ax} = \frac{1}{(D-a)}\,\frac{1}{\phi(a)}\,e^{ax} = \frac{xe^{ax}}{\phi(a)}.$$

If a is a double root of $f(D)=0$, then $D-a$ enters twice as a factor into $f(D)$. Suppose that $f(D)=(D-a)^2\psi(D)$; then

$$\frac{1}{f(D)}e^{ax}=\frac{1}{(D-a)^2}\frac{1}{\psi(D)}e^{ax}=\frac{1}{(D-a)^2}\frac{e^{ax}}{\psi(a)}=\frac{x^2e^{ax}}{2\psi(a)}.$$

The method of procedure is obvious for the case when a is a root of $f(D)=0$, r times.

Ex. 1. Solve $\dfrac{d^3y}{dx^3}+y=3+e^{-x}+5e^{2x}$.

Written in symbolic form, this equation becomes

$$(D^3+1)y=3+e^{-x}+5e^{2x}.$$

Here the roots of $f(D)=0$ are -1, $\dfrac{1\pm i\sqrt{3}}{2}$; hence the complementary function is

$$ce^{-x}+e^{\frac{x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2}+c_2\sin\frac{x\sqrt{3}}{2}\right).$$

The particular integral is

$$\frac{1}{D^3+1}(3e^{0\cdot x}+e^{-x}+5e^{2x});$$

substitution of 0 and 2 for D, on account of the first and third terms, gives $3+\frac{5}{9}e^{2x}$. But -1 is a root of D^3+1, hence, factoring the denominator,

$$\frac{1}{D^3+1}e^{-x}=\frac{1}{D+1}\cdot\frac{1}{D^2-D+1}e^{-x}=\frac{1}{D+1}\cdot\frac{e^{-x}}{3},$$

on substituting -1 for D in the second factor; the last expression is equal to $\dfrac{xe^{-x}}{3}$; hence the complete solution is

$$y=ce^{-x}+e^{\frac{x}{2}}\left(c_1\cos\frac{x\sqrt{3}}{2}+c_2\sin\frac{x\sqrt{3}}{2}\right)+3+\frac{5}{9}e^{2x}+\frac{xe^{-x}}{3}.$$

Ex. 2. Find the particular integrals of Exs. 1, 3, Art. 58, by the short method.

Ex. 3. Solve $\dfrac{d^3y}{dx^3}-y=(e^x+1)^2$.

Ex. 4. Solve $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=3e^{\frac{5}{2}x}$.

61. Integral corresponding to a term of form x^m in the second member, m being a positive integer. When $\dfrac{1}{f(D)}x^m$ is to be evaluated, raise $f(D)$ to the (-1)th power, arranging the terms in ascending powers of D; with the several terms of the expression thus obtained, operate on x^m; the result will be the particular integral corresponding to x^m. It is obvious that terms of the expansion beyond the mth power of D need not be written, since the result of their operation on x^m would be zero.

Ex. 1. Solve $(D^3 + 3D^2 + 2D)y = x^2$.

The roots of $f(D)$ are $0, -2, -1$; and hence the complementary function is $c_1 + c_2e^{-2x} + c_3e^{-x}$.

The particular integral

$$= \frac{1}{2D + 3D^2 + D^3}x^2 = \frac{1}{2D}\left(1 + \tfrac{3}{2}D + \frac{D^2}{2}\right)^{-1}x^2$$

$$= \frac{1}{2D}(1 - \tfrac{3}{2}D + \tfrac{7}{4}D^2 + \cdots)x^2$$

$$= \frac{1}{2D}(x^2 - 3x + \tfrac{7}{2}) = \frac{x}{12}(2x^2 - 9x + 21),$$

$\frac{1}{D}x$ being merely $\int x\,dx$.

The complete solution is $y = c_1 + c_2e^{-2x} + c_3e^{-x} + \frac{x}{12}(2x^2 - 9x + 21)$.

The operator on x^2 could equally well have been put in the form

$$\frac{1}{2}\left(\frac{1}{D} - \frac{3}{2} + \frac{7}{4}D + \cdots\right)x^2;$$

and this gives the result already obtained. One might think that it would be necessary to add another term, D^2, in this form of the operator; but the result for this term would be a numerical constant; and this is already included in the complementary function.

Ex. 2. Solve Ex. 2, Art. 58, by this method.

Ex. 3. Solve $\frac{d^3y}{dx^3} + 8y = x^4 + 2x + 1$.

62. Integral corresponding to a term of form $\sin ax$ or $\cos ax$ in the second member. Successive differentiation of $\sin ax$ gives

$$D \sin ax = a \cos ax,$$

$$D^2 \sin ax = -a^2 \sin ax,$$

$$D^3 \sin ax = -a^3 \cos ax,$$
$$D^4 \sin ax = +a^4 \sin ax = (-a^2)^2 \sin ax,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

and, in general, $(D^2)^n \sin ax = (-a^2)^n \sin ax$.

Therefore, if $\phi(D^2)$ be a rational integral function of D^2,

$$\phi(D^2) \sin ax = \phi(-a^2) \sin ax.$$

From the latter equation and the definition in Art. 57, it follows, since $\phi(-a^2)$ is merely an algebraic multiplier, that

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax.$$

Similarly, it can be shown that

$$\frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax;$$

and, more generally, that

$$\frac{1}{\phi(D^2)} \sin(ax + \alpha) = \frac{1}{\phi(-a^2)} \sin(ax + \alpha),$$

and
$$\frac{1}{\phi(D^2)} \cos(ax + \alpha) = \frac{1}{\phi(-a^2)} \cos(ax + \alpha).$$

Ex. 1. Solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$;

that is, $$(D^3 + D^2 - D - 1)y = \cos 2x.$$

The complementary function is $c_1e^x + e^{-x}(c_2 + c_3x)$; and the particular

integral $$= \frac{1}{D^3 + D^2 - D - 1} \cos 2x = \frac{1}{D+1} \cdot \frac{1}{D^2 - 1} \cos 2x$$
$$= \frac{D-1}{(D^2-1)^2} \cos 2x = \frac{(D-1)}{25} \cos 2x$$
$$= -\frac{2}{25} \sin 2x - \frac{\cos 2x}{25};$$

hence the complete solution is

$$y = c_1e^x + e^{-x}(c_2 + c_3x) - \frac{2}{25} \sin 2x - \frac{\cos 2x}{25}.$$

The number, -4, might have been substituted for D^2 at any step in the work.

Ex. 2. Solve $\frac{d^2y}{dx^2} + a^2y = \cos ax$.

The complementary function is $c_1 \cos ax + c_2 \sin ax$; the particular integral is $\frac{1}{D^2 + a^2} \cos ax = \frac{1}{-a^2 + a^2} \cos ax$; and thus the method fails. In this case, change a to $a + h$; this gives for the value of the particular integral, $\frac{1}{D^2 + a^2} \cos(a + h)x$; this expression, on the application of the principle above and the expansion of the operand by Taylor's series, becomes $\frac{1}{-(a + h)^2 + a^2} (\cos ax - \sin ax \cdot hx - \cos ax \cdot \frac{h^2x^2}{1 \cdot 2 \cdot} + \cdots)$.

The first term is already contained in the complementary function, and hence need not be regarded here; the particular integral will accordingly be written

$$\frac{1}{2a + h}\left(x \sin ax + \frac{hx^2}{1 \cdot 2 \cdot} \cos ax + \text{terms with higher powers of } h\right);$$

on making h approach zero, this reduces to $\frac{x \sin ax}{2a}$.

The complete integral is $y = c_1 \cos ax + c_2 \sin ax + \frac{x \sin ax}{2a}$.

Ex. 3. Solve $\frac{d^2y}{dx^2} - 4y = 2 \sin \frac{1}{2} x$.

Ex. 4. Solve $\frac{d^3y}{dx^3} + y = \sin 3x - \cos^2 \frac{1}{2} x$.

63. Integral corresponding to a term of form $e^{ax}V$ in the second member, V being any function of x.

Since $$De^{ax}V = e^{ax}DV + ae^{ax}V = e^{ax}(D + a)V;$$

and $$D^2e^{ax}V = ae^{ax}(D + a)V + e^{ax}D(D + a)V = e^{ax}(D + a)^2V;$$

and, in general, as is apparent from successive differentiation,

$$D^ne^{ax}V = e^{ax}(D + a)^nV;$$

therefore, $$f(D)e^{ax}V = e^{ax}f(D + a)V. \tag{1}$$

Now put $f(D + a)V = V_1$; then $V = \frac{1}{f(D + a)} V_1$. Also, V_1 will be any function of x, since V is any function of x. Substitution of this value of V in (1) gives

$$f(D)e^{ax} \frac{1}{f(D + a)} V_1 = e^{ax}V_1;$$

whence, operating on both members of this equation with $\dfrac{1}{f(D)}$, and transposing,

$$\frac{1}{f(D)} e^{ax} V_1 = e^{ax} \frac{1}{f(D+a)} V_1,$$

where V_1, as has been observed, is any function of x.

Ex. 1. Solve $\dfrac{d^2y}{dx^2} + y = xe^{2x}$.

The complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{D^2+1} xe^{2x}.$$

By the formula just obtained,

$$\frac{1}{D^2+1} xe^{2x} = e^{2x} \frac{1}{(D+2)^2+1} x = e^{2x} \frac{1}{5 + 4D + D^2} x;$$

and this, by the method of Art. 61, gives the integral $\dfrac{e^{2x}}{25}(5x-4)$.

The complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{e^{2x}}{25}(5x-4).$$

Ex. 2. Solve $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = e^{2x} \sin x$.

Ex. 3. Solve $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$.

64. Integral corresponding to a term of the form xV in the second member, V being any function of x. Suppose that a term of the form xV occurs in $f(D)y = X$.

Differentiation shows that

$$DxV = xDV + V,$$

$$D^2xV = xD^2V + 2DV,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$D^nxV = xD^nV + nD^{n-1}V;$$

or, as it may be written, $\quad = xD^nV + \left(\dfrac{d}{dD} D^n\right) V.$

Therefore, $$f(D)xV = xf(D)V + f'(D)V. \tag{1}$$

The formula in the case of the inverse operator $\frac{1}{f(D)}$ is derived in the following way.

In formula (1) put $f(D)V = V_1$. Then $V = \frac{1}{f(D)}V_1$. Since V is any function of x, V_1 is any function of x. The substitution of this value of V in (1) gives

$$f(D)\,x\frac{1}{f(D)}V_1 = xV_1 + f'(D)\frac{1}{f(D)}V_1.$$

On operating on both members of this equation with $\frac{1}{f(D)}$, and transposing,

$$\frac{1}{f(D)}xV_1 = x\frac{1}{f(D)}V_1 - \frac{1}{f(D)}\cdot f'(D)\cdot\frac{1}{f(D)}V_1$$

$$= \left\{x - \frac{1}{f(D)}\cdot f'(D)\right\}\frac{1}{f(D)}V_1.$$

The particular integral corresponding to expressions of the form x^rV, where r is a positive integer, can be obtained by successive applications of this method. Constants of integration should not be introduced in the process of finding particular integrals.

Ex. 1. Find the particular integral of Ex. 1, Art. 63, by this method.

$$\text{The particular integral} = \frac{1}{D^2+1}xe^{2x} = \left(x - \frac{1}{D^2+1}\cdot 2D\right)\frac{1}{D^2+1}e^{2x}$$

$$= \frac{xe^{2x}}{5} - \frac{1}{D^2+1}\cdot 2D\frac{e^{2x}}{5}$$

$$= \frac{xe^{2x}}{5} - \frac{1}{D^2+1}\frac{4e^{2x}}{5}$$

$$= \frac{e^{2x}}{25}(5x-4).$$

Ex. 2. Solve $\frac{d^2y}{dx^2} + 4y = x\sin x$. Ex. 3. Solve $\frac{d^2y}{dx^2} - y = x^2\cos x$.

EXAMPLES ON CHAPTER VI.

1. $\frac{d^4y}{dx^4} + 4y = 0$.
2. $(D^5 - 13D^3 + 26D^2 + 82D + 104)y = 0$.

3. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^2 + x.$
4. $\frac{d^2y}{dx^2} + 4y = \sin 3x + e^x + x^2.$
5. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x + e^{mx}.$
6. $(D^2 - a^2)y = e^{ax} + e^{nx}.$
7. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = x.$
8. $\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = x^2(a + bx).$
9. $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} + 12y = x.$
10. $\frac{d^4y}{dx^4} + 2n^2\frac{d^2y}{dx^2} + n^4y = \cos mx.$
11. $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x.$
12. $\frac{d^2y}{dx^2} + a^2y = \sec ax.$
13. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^{3x}.$
14. $(D^2 + n^2)y = e^x x^4.$
15. $\frac{d^4y}{dx^4} - a^4y = x^4.$
16. $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = x.$
17. $\frac{d^4y}{dx^4} - y = e^x \cos x.$
18. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$
19. $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = e^{2x}(1 + x).$
20. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x.$
21. $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-x}.$
22. $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = x^2e^x.$
23. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = xe^x + e^x.$
24. $\frac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)e^x.$
25. $(D^2 - 4D + 3)y = e^x \cos 2x + \cos 3x.$
26. $(D^3 - 3D^2 + 4D - 2)y = e^x + \cos x.$
27. $(D^2 - 9D + 20)y = 20x.$
28. $(D^3 - 3D^2 + 4)y = e^{3x}.$
29. $\frac{d^3y}{dx^3} + y = e^{2x} \sin x + e^{\frac{x}{2}} \sin \frac{x\sqrt{3}}{2}.$

30. Show that $\frac{ND + M}{(D - \alpha)^2 + \beta^2}X$, where N and M are constants, is equal to twice the real part of $\frac{1}{2i\beta}\left(\frac{1}{D - \alpha - i\beta}\right)X$ operated on by $ND + M$; that is, to $\frac{e^{\alpha x}\sin \beta x}{\beta}\int e^{-\alpha x}\cos\beta x \cdot X dx - \frac{e^{\alpha x}\cos\beta x}{\beta}\int e^{-\alpha x}\sin \beta x \cdot X dx.$

(Johnson, *Diff. Eq.*, Art. 106.)

CHAPTER VII.

LINEAR EQUATIONS WITH VARIABLE COEFFICIENTS.

65. The homogeneous linear equation. First method of solution. This chapter will treat of linear equations in which the coefficients are functions of x. For the most part, however, it will discuss only a very special class of these equations, namely, the homogeneous linear equation.

A homogeneous linear equation is an equation of the form

$$x^n\frac{d^ny}{dx^n}+p_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_{n-1}x\frac{dy}{dx}+p_ny=X, \tag{1}$$

where p_1, p_2, $\cdots$, p_n, are constants, and X is a function of x.

This equation can be transformed into an equation with constant coefficients by changing the independent variable x to z, the relation between x and z being

$$z=\log x, \text{ that is, } x=e^z.$$

If this change be made, then

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz},$$

$$\frac{d^2y}{dx^2}=\frac{1}{x^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right),$$

$$\frac{d^3y}{dx^3}=\frac{1}{x^3}\left(\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}\right),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\frac{d^ny}{dx^n}=\frac{1}{x^n}\left(\frac{d^ny}{dz^n}-\frac{n(n-1)}{2}\frac{d^{n-1}y}{dz^{n-1}}+\cdots+(-1)^{n-1}\underline{|n-1}\frac{dy}{dz}\right)$$

On putting D for $\dfrac{d}{dz}$, and clearing of fractions, these equations become

$$\left.\begin{aligned} x\frac{dy}{dx} &= Dy, \\ x^2\frac{d^2y}{dx^2} &= D(D-1)y, \\ x^3\frac{d^3y}{dx^3} &= D(D-1)(D-2)y, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x^n\frac{d^ny}{dx^n} &= D(D-1)(D-2)\cdots(D-n+1)y; \end{aligned}\right\} \tag{2}$$

and, in general, the operators only and not the operand being indicated,

$$x^r\frac{d^r}{dx^r} = D(D-1)\cdots(D-r+1). \tag{3}$$

Hence, the substitution of e^z for x in (1) changes it into the form

$$\{D(D-1)\cdots(D-n+1)+p_1D(D-1)\cdots(D-n+2)+\cdots+p_n\}y \\ = Z, \tag{4}$$

where Z is the function of z into which X is changed.

Equation (4), having constant coefficients, can be solved by the methods of the last chapter. If its solution be $y=f(z)$, then the solution of (1) is $y=f(\log x)$.

Therefore equation (1) can be solved by putting x equal to e^z, thus changing the independent variable from x to z, which reduces the given equation to one with constant coefficients; and then solving the newly formed equation by the methods of the preceding chapter.

Ex. 1. Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2\log x$.

On changing the independent variable by putting x equal to e^z, this equation becomes

$$\{D(D-1)-D+1\}y = 2z.$$

On solving this equation by the methods of Arts. 51, 61, the complete integral is found to be

$$y = e^z(c_1 + c_2 z) + 2z + 4,$$

which, in terms of x, is

$$y = x(c_1 + c_2 \log x) + 2 \log x + 4.$$

Ex. 2. Solve $x^2\dfrac{d^2y}{dx^2} + y = 3x^2$.

66. Second method of solution: (A) To find the complementary function. The solution of

$$x^n\frac{d^ny}{dx^n} + p_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_ny = X \tag{1}$$

can be found directly, without explicitly making the transformation shown in Art. 65.

The first member of (1) becomes, when x^m is substituted for y,

$$\{m(m-1)(m-2)\cdots(m-n+1) + p_1m(m-1)\cdots(m-n+2) \\ + \cdots + p_n\}x^m.$$

Therefore, if

$$m(m-1)\cdots(m-n+1) + p_1m(m-1)\cdots(m-n+2) \\ + \cdots + p_n = 0, \tag{2}$$

the substitution of x^m for y makes the first member of (1) vanish; and then x^m is a part of the complementary function of the solution of (1). Therefore, if the n roots of (2) are $m_1, m_2, \cdots, m_n$, the complementary function of the solution of (1) is

$$c_1x^{m_1} + c_2x^{m_2} + \cdots + c_nx^{m_n},$$

the c's being arbitrary constants.

It will be noticed that the first member of (2) is the same function of m as the coefficient of y in equation (4) Art. 65 is of D. Therefore, corresponding to an integral $y = x^{m_1}$ of (1), there is an integral $y = e^{m_1z}$ of (4) Art. 65; and hence, as has already been seen, an integral of (1) can be obtained by substituting $\log x$ for z in the integral of (4) Art. 65. Therefore, if (2) has a root m_1 repeated r times, the corresponding integral of equation (4) Art. 65 being

$$y = e^{m_1 z}(c_1 + c_2 z + c_3 z^2 + \cdots + c_r z^{r-1}),$$

the integral of (1) is

$$y = x^{m_1}\{c_1 + c_2 \log x + \cdots + c_r (\log x)^{r-1}\}.$$

Similarly, if (2) have a pair of imaginary roots $\alpha \pm i\beta$, the corresponding integral of (4) Art. 65 being

$$y = e^{\alpha z}(c_1 \cos \beta z + c_2 \sin \beta z),$$

the integral of (1) is

$$y = x^{\alpha}\{c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)\}.$$

Ex. 1. Solve $x^3\dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0.$

Substitution of x^m for y gives

$$(m^3 + 1)x^m = 0;$$

the roots of this equation are -1, $\dfrac{1 \pm \sqrt{3}\,i}{2}$;

and hence the solution is

$$y = \frac{c_1}{x} + x^{\frac{1}{2}}\left\{c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right\}.$$

Ex. 2. Find the complementary functions of Exs. 1, 2, Art. 65, by this method.

Ex. 3. Solve $x^4\dfrac{d^4y}{dx^4} + 6x^3\dfrac{d^3y}{dx^3} + 9x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = 0.$

67. Second method of solution: (B) To find the particular integral. In this article and the two following, the particular integral of

$$x^n\frac{d^ny}{dx^n} + p_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_n y = X \qquad (1)$$

will be found.

Since the symbol D in (3) Art. 65 stands for $\frac{d}{dz}$, that is, for $x\frac{d}{dx}$, this equation may be written

$$x^r\frac{d^r}{dx^r} = x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)\cdots\left(x\frac{d}{dx}-r+1\right);^*$$

therefore (1), when θ is substituted for $x\frac{d}{dx}$ therein,† takes the form

$$\{\theta(\theta-1)\cdots(\theta-n+1)+p_1\theta(\theta-1)\cdots(\theta-n+2)+\cdots+p_n\}y=X.$$

The coefficient of y here is the same function of θ as the first member of (2) Art. 66 is of m.

Let this equation be written in the symbolic form

$$f(\theta)\,y = X. \tag{3}$$

The method deduced in Art. 66 for finding the complementary function of (3) may on making use of the symbol θ be thus indicated: If the n roots of $f(\theta)=0$ are $\theta_1, \theta_2, \cdots, \theta_n$, the complementary function of (3) is

$$c_1x^{\theta_1}+c_2x^{\theta_2}+\cdots+c_nx^{\theta_n},$$

the c's being arbitrary constants.

The particular integral of (3), after the manner used in the case of $f(D)\,y=X$, in the last chapter, may be expressed in the form $\frac{1}{f(\theta)}X$. A method for evaluating $\frac{1}{f(\theta)}X$ must now be devised.

68. The symbolic functions $f(\theta)$ and $\frac{1}{f(\theta)}$. As to the direct symbol $f(\theta)$, it is to be observed that its factors are commutative. For example,

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* See Forsyth, *Differential Equations*, Arts. 36, 37.

† Thus θ stands for the $\frac{d}{dz}$ of Art. 65, which was there symbolized by D; but as D had already been used to indicate $\frac{d}{dx}$, this new symbol is required.

$$(\theta-\alpha)(\theta-\beta)y = x^2\frac{d^2y}{dx^2}-(\alpha+\beta-1)x\frac{dy}{dx}+\alpha\beta y;$$

and
$$(\theta-\alpha)(\theta-\beta)(\theta-\gamma)y = x^3\frac{d^3y}{dx^3}+(3-\alpha-\beta-\gamma)x^2\frac{d^2y}{dx^2}$$
$$+(\alpha\beta+\beta\gamma+\gamma\alpha-\alpha-\beta-\gamma+1)x\frac{dy}{dx}-\alpha\beta\gamma y;$$

and this shows, by the symmetry of the constant coefficients, that the order of the operative factors is indifferent. The student can complete the proof of this theorem concerning $f(\theta)$ for himself.

If $\frac{1}{f(\theta)}X$ be defined as that function of x which when operated upon by $f(\theta)$ will give X, then it can be shown, by the method followed in Art. 57 in the case of the symbolic function $\frac{1}{f(D)}$, that $\frac{1}{f(\theta)}$ can be decomposed into factors which are commutative; and also that it can be broken up into partial fractions.

69. Methods of finding the particular integral. It is thus apparent that the particular integral of (3) Art. 67 can be found in the following ways:

(*a*) The operator $\frac{1}{f(\theta)}$ may be expressed in factorial form, and $\frac{1}{f(\theta)}X$ will then become

$$\frac{1}{\theta-\alpha_1}\cdot\frac{1}{\theta-\alpha_2}\cdots\frac{1}{\theta-\alpha_n}X;$$

here the operations indicated by the factors are to be taken in succession, beginning with the first on the right; the final result will be the particular integral.

(*b*) The operator $\frac{1}{f(\theta)}$ may be broken up into partial fractions, and consequently $\frac{1}{f(\theta)}X$ be thus expressed,

$$\left(\frac{N_1}{\theta-\alpha_1}+\frac{N_2}{\theta-\alpha_2}+\cdots+\frac{N_n}{\theta-\alpha_n}\right)X;$$

the sum of the results of the operations indicated will be the particular integral.

Since the methods (a) and (b) are made up of operations of the kind effected by $\frac{1}{\theta-\alpha}$ upon X, the result of the latter operation should be impressed upon the memory.

Now, $\frac{1}{\theta-\alpha}X$ is the particular integral of the linear equation of the first order

$$x\frac{dy}{dx}-\alpha y=X.$$

The particular integral of this equation, by Art. 20, is found to be $x^{\alpha}\int x^{-\alpha-1}X\,dx$; therefore

$$\frac{1}{\theta-\alpha}X=x^{\alpha}\int x^{-\alpha-1}X\,dx.$$

Hence the method (a) will give

$$x^{\alpha_1}\int x^{\alpha_2-\alpha_1-1}\int\cdots\int x^{-\alpha_n-1}X(dx)^n$$

as the value of the particular integral of (3) Art. 67; and the method (b) will give

$$N_1x^{\alpha_1}\int x^{-\alpha_1-1}X\,dx+N_2x^{\alpha_2}\int x^{-\alpha_2-1}X\,dx+\cdots+N_nx^{\alpha_n}\int x^{-\alpha_n-1}X\,dx$$

as its value.

When a term $\frac{N}{(\theta-\alpha)^m}$ is one of the partial fractions of (b), the operator $\frac{1}{\theta-\alpha}$ must be applied to X, m times in succession; and this will give the result

$$x^{\alpha}\int x^{-1}\int x^{-1}\int\cdots\int x^{-\alpha-1}X\,(dx)^m.$$

Ex. 1. Solve $x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$.

Here $f(\theta)$ is $\theta(\theta-1)-2\theta-4$, which reduces to $(\theta-4)(\theta+1)$.

Hence the complementary function is $c_1x^4 + \frac{c_2}{x}$, and the particular integral is $\frac{1}{(\theta - 4)(\theta + 1)}x^4$.

On using partial fractions, the latter will be written

$$\frac{1}{5}\left(\frac{1}{\theta - 4}x^4 - \frac{1}{\theta + 1}x^4\right); \text{ this reduces to}$$

$$\tfrac{1}{5}x^4\int x^{-4\;\;1+4}dx - \tfrac{1}{5}x^{-1}\int x^{1-1+4}dx, \text{ which on integration is}$$

$$\frac{x^4\log x}{5} - \frac{x^4}{25}.$$

The complete integral is, therefore,

$$y = c_1x^4 + \frac{c_2}{x} + \frac{x^4\log x}{5};$$

the term $-\frac{x^4}{25}$ is included in the term c_1x^4 of the complementary function

Ex. 2. Solve $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = x^4$.

Ex. 3. Solve $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 20y = (x + 1)^2$.

70. Integral corresponding to a term of form x^a in the second member. In the case of the homogeneous linear equation, as in that of the equation discussed in the last chapter, methods shorter than the general one can be deduced for finding the particular integrals which correspond to terms of special form in the second member. For instance,

$$\frac{1}{f(\theta)}x^m = \frac{1}{f(m)}x^m.$$

PROOF: $\left(x\frac{d}{dx}\right)x^m = mx^m$, $\left(x\frac{d}{dx}\right)^2x^m = m^2x^m$, and for any positive integer r, $\left(x\frac{d}{dx}\right)^r x^m = m^rx^m$. Now $f(\theta)$ is a rational integral function of θ, and therefore

$$f(\theta)x^m = f(m)x^m.$$

Applying $\frac{1}{f(\theta)}$ to both members of this equation, and transposing, $f(m)$ being merely an algebraic multiplier,

$$\frac{1}{f(\theta)}x^m = \frac{1}{f(m)}x^m.$$

If m is a root of $f(\theta)=0$, then $f(m)=0$; and hence the method fails. In this case $f(\theta)$ can be factored into $(\theta-m)\phi(\theta)$; the particular integral then becomes $\frac{1}{\theta-m}\frac{1}{\phi(\theta)}x^m$, which reduces to $\frac{1}{\phi(m)}\cdot\frac{1}{\theta-m}x^m$; this is equal to $\frac{x^m \log x}{\phi(m)}$.

If m is repeated as a root r times, $f(\theta)=F(\theta)(\theta-m)^r$, and the corresponding integral is $\frac{1}{F(m)}\frac{1}{(\theta-m)^r}x^m$, which has the value $\frac{x^m(\log x)^r}{r!\,F(m)}$.

Ex. 1. Solve $x^2\frac{d^2y}{dx^2}+7x\frac{dy}{dx}+5y=x^5$.

Here $f(\theta)$ is $\theta^2+6\theta+5$;

hence the complementary function is $c_1x^{-1}+c_2x^{-5}$; and the particular integral is $\frac{1}{\theta^2+6\theta+5}x^5$, which, on substituting 5 for θ, becomes $\frac{x^5}{60}$. The complete solution is thus

$$y=c_1x^{-3}+c_2x^{-2}+\frac{x^5}{60}.$$

Ex. 2. Find the particular integrals of Exs. 1, 2, 3, Art. 69 by this method.

71. Equations reducible to the homogeneous linear form. There are some equations that are easily reducible to the homogeneous linear form; and hence also to the form of the linear equation with constant coefficients; for, as has been seen in Art. 65, these two forms are transformable into each other.

Any equation of the form *

$$(a+bx)^n\frac{d^ny}{dx^n}+P_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots$$
$$+P_{n-1}(a+bx)\frac{dy}{dx}+P_ny=F(x), \quad (1)$$

where the coefficients $P_1, \ldots, P_n$, are constants, is transformed into the homogeneous linear equation,

$$z^n\frac{d^ny}{dz^n}+\frac{P_1}{b}z^{n-1}\frac{d^{n-1}y}{dz^{n-1}}+\frac{P_2}{b^2}z^{n-2}\frac{d^{n-2}y}{dz^{n-2}}+\cdots$$
$$+\frac{P_{n-1}}{b^{n-1}}z\frac{dy}{dz}+\frac{P_n}{b^n}y=\frac{1}{b^n}F\left(\frac{z-a}{b}\right), \quad (2)$$

---

* This is known as Legendre's equation. See footnote, p. 105.

when the independent variable is changed from x to z, by the substitution of z for $a + bx$.

If the solution of (2) be $y = F(z)$, the solution of (1) is $y = F(a + bx)$.

If e^t had been substituted for $a + bx$, the independent variable thus being changed from x to t, there would have been derived a linear equation with constant coefficients.

Ex. 1. Solve $(5 + 2x)^2 \dfrac{d^2y}{dx^2} - 6(5 + 2x)\dfrac{dy}{dx} + 8y = 0.$

Ex. 2. Solve $(2x - 1)^3 \dfrac{d^3y}{dx^3} + (2x - 1)\dfrac{dy}{dx} - 2y = 0.$

EXAMPLES ON CHAPTER VII.

1. $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2y}{x^3} = 1.$

2. $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x.$

3. $x^3\dfrac{d^3y}{dx^3} - 3x^2\dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} - 8y = 0.$

4. $(x + a)^2\dfrac{d^2y}{dx^2} - 4(x + a)\dfrac{dy}{dx} + 6y = x.$

5. $x^3\dfrac{d^3y}{dx^3} + 6x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} - 4y = 0.$

6. $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right).$

7. $16(x + 1)^4\dfrac{d^4y}{dx^4} + 96(x + 1)^3\dfrac{d^3y}{dx^3} + 104(x + 1)^2\dfrac{d^2y}{dx^2} + 8(x + 1)\dfrac{dy}{dx} + y$
$= x^2 + 4x + 3.$

8. $(x^2D^2 + xD - 1)y = x^m.$

9. $(x^2D^2 - 3xD + 4)y = x^m.$

10. $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \log x)^2.$

11. $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1.$

12. $(x^2D^2 + 3xD + 1)y = \dfrac{1}{(1 - x)^2}.$

13. $[x^2D^2 - (2m - 1)xD + (m^2 + n^2)]y = n^2x^m \log x.$

14. $x^2\dfrac{dy^2}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{\log x \cdot \sin(\log x) + 1}{x}.$

CHAPTER VIII.

EXACT DIFFERENTIAL EQUATIONS, AND EQUATIONS OF PARTICULAR FORMS. INTEGRATION IN SERIES.

72. In this chapter linear differential equations that are exact will be first discussed, and then equations of certain particular forms will be considered. Some of the latter come under some one of the types already treated; but in obtaining their solutions, special modifications of the general methods can be employed. It often happens that an equation at the same time belongs to several forms; instances of this will be found among the examples in Arts. 76, 77, 78, 79, 80.

73. Exact differential equations defined. A differential equation,

$$f\left(\frac{d^ny}{dx^n}, \cdots, \frac{dy}{dx}, y\right) = X,$$

is said to be exact when it can be derived by differentiation merely, and without any further process, from an equation of the next lower order

$$f\left(\frac{d^{n-1}y}{dx^{n-1}}, \cdots, \frac{dy}{dx}, y\right) = \int X dx + c.$$

Exact differential equations of the first order have been treated in Art. 11.

74. Criterion of an exact differential equation. The condition will now be found which the coefficients of a differential equation,

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_ny = 0, \tag{1}$$

must satisfy in order that it be exact. The coefficients P_0, $P_1, \cdots, P_n$, are functions of x; in what follows, their successive derivatives will be indicated by P', P'', $\cdots$, $P^{(n)}$.

The first term of (1) is evidently derivable by direct differentiation from $P_0\dfrac{d^{n-1}y}{dx^{n-1}}$, which is therefore the first term of the integral of (1); on differentiating this and subtracting the result from the first member of (1), there remains

$$(P_1 - P_0')\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_ny. \tag{2}$$

The first term of (2) is evidently derivable by differentiation from $(P_1 - P_0')\dfrac{d^{n-2}y}{dx^{n-2}}$, which is therefore the second term of the integral of (1). On subtracting the derivative of this term from (2) there remains

$$(P_2 - P_1' + P_0'')\frac{d^{n-2}y}{dx^{n-2}} + P_3\frac{d^{n-3}y}{dx^{n-3}} + \cdots + P_ny.$$

The first term of this expression is derivable from

$$(P_2 - P_1' + P_0'')\frac{d^{n-3}y}{dx^{n-3}},$$

which will therefore be the third term of the integral of (1). By continuing this process, the expression

$$\{P_{n-1} - P_{n-2}' + \cdots + (-1)^{n-1}P_0^{(n-1)}\}\frac{dy}{dx} + P_ny \tag{3}$$

will be reached, the first term of which is evidently derivable from

$$\{P_{n-1} - P_{n-2}' + \cdots + (-1)^{n-1}P_0^{(n-1)}\}y. \tag{4}$$

Both terms of (3) will be derived from the expression (4), if the derivative of the coefficient of (y) in (4) be equal to P_n, that is, if

$$P_n - P_{n-1}' + \cdots + (-1)^nP_0^{(n)} = 0. \tag{5}$$

But if both terms of (3) are derivable from the expression (4), an integral of (1) has now been obtained in the form

$$P_0 \frac{d^{n-1}y}{dx^{n-1}} + (P_1 - P_0') \frac{d^{n-2}y}{dx^{n-2}} + (P_2 - P_1' + P_0'') \frac{d^{n-3}y}{dx^{n-3}}$$
$$+ \cdots + \{P_{n-1} - P_{n-2}' + \cdots (-1)^n P_0^{(n-1)}\} y = c_1. \qquad (6)$$

Therefore (1) *has an integral* (6), that is, (1) *is exact if its coefficients satisfy condition* (5).

75. The integration of an exact equation; first integrals. If the second member of (1) Art. 74 be $f(x)$, and condition (5) be satisfied, the second member of (6) will be $\int f(x)\,dx + c$.

If equation (6) Art. 74 is also exact its integral can be found in the same way; and the process can be repeated until an equation of the second or higher order that is not exact is reached; in some cases, this process can be carried on until a value of $\frac{dy}{dx}$, or of y is obtained. Equation (6) is called a *first integral* of (1) Art. 74.

It is easily proven by means of Arts. 3, 4, and Note C, p. 194, that any equation of the nth order has exactly n independent first integrals.* [See note, page 108.]

Sometimes equations that are not linear can be solved by the 'trial method' employed in the case of (1) Art. 74, for instance, Exs. 6, 7, below.

Ex. 1. Solve $x\frac{d^3y}{dx^3} + (x^2 - 3)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0.$

This is neither a homogeneous linear equation, nor one having constant coefficients. In it, $P_0 = x$, $P_1 = x^2 - 3$, $P_2 = 4x$, $P_3 = 2$; and the condition that it be exact is satisfied by these values.

Integration gives

$$x\frac{d^2y}{dx^2} + (x^2 - 4)\frac{dy}{dx} + 2xy = c_1.$$

The condition under which this equation is exact is also satisfied; integrating again,

$$x\frac{dy}{dx} + (x^2 - 5)y = c_1 x + c_2.$$

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* See Forsyth, *Differential Equations*, Arts. 7, 8.

This is not exact, but it is a linear equation of the first order, and hence solvable by the method of Art. 20. The solution is

$$\frac{e^{\frac{x^2}{2}}y}{x^5} = c_1\int\frac{e^{\frac{x^2}{2}}}{x^5}dx + c_2\int\frac{e^{\frac{x^2}{2}}}{x^6}dx + c_3.$$

Ex. 2. Solve $x^5\dfrac{d^2y}{dx^2} + 3x^3\dfrac{dy}{dx} + (3-6x)x^2y = x^4 + 2x - 5.$

The coefficients of this equation do not satisfy condition (5) Art. 74; and hence it is not exact. However, an integrating factor x^m can be deduced, which will render it an exact differential equation.

Multiplication by x^m gives

$$x^{5+m}\frac{d^2y}{dx^2} + 3x^{3+m}\frac{dy}{dx} + (3-6x)x^{2+m}y = (x^4 + 2x - 5)x^m.$$

Application of condition (5) Art. 74 to the first member, shows that for that condition to hold, m must satisfy the equation

$$(m+2)(m+7)x^{m+3} - 3(m+2)x^{m+2} = 0;$$

that is, m must be equal to -2.

On using the factor $\dfrac{1}{x^2}$, the original equation becomes

$$x^3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (3-6x)y = x^2 + \frac{2}{x} - \frac{5}{x^2},$$

which is exact.

Integration gives the first integral

$$x^3\frac{dy}{dx} + 3x(1-x)y = \frac{x^3}{3} + 2\log x + \frac{5}{x},$$

a linear equation of the first order.

Ex. 3. Solve $x\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + 2y = 0.$

Ex. 4. Solve $\dfrac{d^2y}{dx^2} + 2e^x\dfrac{dy}{dx} + 2e^xy = x^2.$

Ex. 5. Solve $\sqrt{x}\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} + 3y = x.$

Ex. 6. Find a first integral of $\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} - x^2y\dfrac{dy}{dx} = xy^2.$

Ex. 7. Solve $x^2y\dfrac{d^2y}{dx^2} + \left(x\dfrac{dy}{dx} - y\right)^2 - 3y^2 = 0.$

76. Equations of the form $\frac{d^ny}{dx^n}=f(x)$.

This is an exact differential equation. Integration gives $\frac{d^{n-1}y}{dx^{n-1}}=\int f(x)\,dx+c_1$; a second integration gives

$$\frac{d^{n-2}y}{dx^{n-2}}=\int\int f(x)(dx)^2+c_1x+c_2;$$

by proceding in this way, the complete integral

$$y=\int\int\cdots\int f(x)(dx)^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n$$

is obtained.

Ex. 1. Solve $\frac{d^3y}{dx^3}=xe^x$.

Integrating, $\frac{d^2y}{dx^2}=xe^x-e^x+c_1$,

$$\frac{dy}{dx}=xe^x-2e^x+c_1x+c_2,$$

$$y=xe^x-3e^x+cx^2+c_2x+c_3.$$

This equation could also have been solved by the method of Chap. VI.

Ex. 2. Solve $\frac{d^ny}{dx^n}=x^m$.

Ex. 3. Solve $x^2\frac{d^4y}{dx^4}+1=0$.

Ex. 4. Solve $\frac{d^2y}{dx^2}=x^2\sin x$.

77. Equations of the form $\frac{d^2y}{dx^2}=f(y)$.

An equation of the form

$$\frac{d^2y}{dx^2}=f(y),$$

which in general is not linear and is not an exact differential equation, can be solved in the following way:

Multiplication of both sides by $2\frac{dy}{dx}$ gives

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}=2f(y)\frac{dy}{dx};$$

integrating,
$$\left(\frac{dy}{dx}\right)^2 = 2\int f(y)\,dy + c_1.$$

From this,
$$\frac{dy}{\left\{2\int f(y)\,dy + c_1\right\}^{\frac{1}{2}}} = dx,$$

whence
$$\int \frac{dy}{\left\{2\int f(y)\,dy + c_1\right\}^{\frac{1}{2}}} = x + c_2.$$

Ex. 1. Solve $\dfrac{d^2y}{dx^2} + a^2y = 0.$

Multiplying by $2\dfrac{dy}{dx}$, $\quad 2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = -2a^2y\dfrac{dy}{dx};$

integrating, $\quad \left(\dfrac{dy}{dx}\right)^2 = -a^2y^2 + c_1 = a^2(c^2 - y^2)$

on putting a^2c^2 for c_1;

separating the variables, $\quad \dfrac{dy}{\sqrt{c^2 - y^2}} = a\,dx,$

and integrating, $\quad \sin^{-1}\dfrac{y}{c} = ax + c_2,$

hence, $\quad y = c\sin(ax + c_2).$

The given differential equation is linear, and y can be obtained directly by the method of Art. 52. The roots of the auxiliary equation being $\pm ia$, the solution is
$$y = c_1 \sin ax + c_2 \cos ax;$$

that is, $\quad y = c_1 \sin(ax + c_3).$

This equation is an important one in physical applications.

Ex. 2. Solve $\dfrac{d^2y}{dx^2} = \dfrac{1}{\sqrt{ay}}.$

Ex. 3. Solve $\dfrac{d^2y}{dx^2} + \dfrac{a^2}{y^2} = 0.$

Ex. 4. Solve $\dfrac{d^2y}{dx^2} - a^2y = 0.$

78. Equations that do not contain y directly. The typical form of these equations is
$$f\left(\frac{d^ny}{dx^n}, \cdots, \frac{dy}{dx}, x\right) = 0. \tag{1}$$

Equations of this kind of the first order were considered in Art. 26; and those of Art. 76 also belong to this class.

If p be substituted for $\frac{dy}{dx}$, then $\frac{d^2y}{dx^2}=\frac{dp}{dx}, \cdots, \frac{d^ny}{dx^n}=\frac{d^{n-1}p}{dx^{n-1}}$; and (1) takes the form

$$f\left(\frac{d^{n-1}p}{dx^{n-1}}, \cdots, p, x\right)=0,$$

an equation of the $(n-1)$th order between x and p; and this may possibly be solved for p. Suppose that the solution is

$$p=\frac{dy}{dx}=F(x);$$

then
$$y=\int F(x)\,dx+c.$$

If the derivative of lowest order appearing in the equation be $\frac{d^ry}{dx^r}$, put $\frac{d^ry}{dx^r}=p$, find p, and therefrom find y by Art. 76.

Ex. 1. Solve $x^2\frac{d^3y}{dx^3}-4x\frac{d^2y}{dx^2}+6\frac{dy}{dx}=4.$

Putting p for $\frac{dy}{dx}$, this becomes

$$x^2\frac{d^2p}{dx^2}-4x\frac{dp}{dx}+6p=4;$$

integrating,
$$p=c_1x^2+c_2x^3+\tfrac{2}{3},$$

whence
$$y=ax^3+bx^4+\tfrac{2}{3}x+c.$$

Ex. 2. Solve $\frac{d^2y}{dx^2}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$

Ex. 3. Solve $(1+x^2)\frac{d^2y}{dx^2}+1+\left(\frac{dy}{dx}\right)^2=0.$

Ex. 4. Solve $2x\frac{d^3y}{dx^3}\frac{d^2y}{dx^2}=\left(\frac{d^2y}{dx^2}\right)^2-a^2.$

79. Equations that do not contain x directly. The typical form of these equations is

$$f\left(\frac{d^ny}{dx^n}, \cdots, \frac{dy}{dx}, y\right)=0. \qquad (1)$$

Equations of this kind of the first order were considered in Art. 26; and those of Art. 77 also belong to this class.

If p be substituted for $\frac{dy}{dx}$, then

$$\frac{d^2y}{dx^2}=p\frac{dp}{dy},\ \frac{d^3y}{dx^3}=p^2\frac{d^2p}{dy^2}+p\left(\frac{dp}{dy}\right)^2, \text{ etc.};$$

and (1) will take the form

$$f\left(\frac{d^{n-1}p}{dy^{n-1}},\ \cdots,\ p,\ y\right)=0,$$

an equation of the $(n-1)$th order between y and p which may possibly be solved for p. Suppose that the solution is

$$p=f(y);$$

then the solution of (1) is

$$\int\frac{dy}{f(y)}=x+c.$$

Ex. 1. Solve $\frac{d^2y}{dx^2}-a\left(\frac{dy}{dx}\right)^2=0.$

Ex. 2. Solve $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=1.$

Ex. 3. Solve $y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2=y^2\log y.$

Ex. 4. Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+4\left(\frac{dy}{dx}\right)^3=0$ by the method of this article, and by that of Art. 78.

80. Equations in which y appears in only two derivatives whose orders differ by two. The typical form of these equations is

$$f\left(\frac{d^ny}{dx^n},\ \frac{d^{n-2}y}{dx^{n-2}},\ x\right)=0. \tag{1}$$

If q be substituted for $\frac{d^{n-2}y}{dx^{n-2}}$, then $\frac{d^ny}{dx^n}=\frac{d^2q}{dx^2}$; and (1) becomes

$$f\left(\frac{d^2q}{dx^2}, q, x\right)=0;$$

from which q, that is, $\frac{d^{n-2}y}{dx^{n-2}}$, may be found. Suppose that the solution is

$$q=\frac{d^{n-2}y}{dx^{n-2}}=\phi(x),$$

then y can be found by the method of successive integration shown in Art. 76.

Ex. 1. Solve $\frac{d^4y}{dx^4}+a^2\frac{d^2y}{dx^2}=0$ both by this method and that of Chap. VI.

Ex. 2. Solve $\frac{d^5y}{dx^5}-m^2\frac{d^3y}{dx^3}=e^{ax}$ both by this method and that of Chap. VI.

Ex. 3. Solve $x^2\frac{d^4y}{dx^4}+a^2\frac{d^2y}{dx^2}=0$.

81. Equations in which y appears in only two derivatives whose orders differ by unity. The typical form of these equations is

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, x\right)=0. \tag{1}$$

If q be substituted for $\frac{d^{n-1}y}{dx^{n-1}}$, then $\frac{d^ny}{dx^n}=\frac{dq}{dx}$; and (1) becomes

$$f\left(\frac{dq}{dx}, q, x\right)=0,$$

an equation of the first order between q and x; its solution will give the value of q in terms of x. Suppose that the solution is

$$q=\frac{d^{n-1}y}{dx^{n-1}}=F(x);$$

then, from this relation, by successive integration, the value of y can be deduced.

Ex. 1. Solve $a^2\frac{d^2y}{dx^2}\frac{dy}{dx}=x.$

On putting q for $\frac{dy}{dx}$, this becomes

$$a^2q\frac{dq}{dx}=x;$$

integrating, $$aq=a\frac{dy}{dx}=\sqrt{x^2+c^2};$$

integrating again, $ay=\frac{1}{2}[x\sqrt{x^2+c^2}+c^2\log(x+\sqrt{x^2+c^2})+c_2].$

Ex. 2. Solve $a\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}.$

Ex. 3. Solve $x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0.$

Ex. 4. Solve $\frac{d^3y}{dx^3}\cdot\frac{d^2y}{dx^2}=2.$

82. Integration of linear equations in series. When an equation belongs to a form which cannot be solved by any of the methods hitherto discussed, recourse may be had to finding a convergent series arranged according to powers of the independent variable, which will approximately express the value of the dependent variable. For the purposes of this article it is assumed that such a series can be obtained.*

Suppose that the linear equation

$$P_0\frac{d^ny}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+P_ny=0 \tag{1}$$

can have a solution of the form

$$y=A_0x^m+A_1x^{m_1}+A_2x^{m_2}+\cdots+A_rx^{m_r}+\cdots, \tag{2}$$

where the second member is a finite sum or a convergent series for some value or values of x. Concerning this series three things must be known: namely, the initial term, the relation between the exponents of x, and the relation between the coefficients.

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* See Note B. Also Forsyth, *Differential Equations*, Arts. 83, 84.

The following examples show how these things can be determined:*

Ex. 1. Solve $(x-x^2)\frac{d^2y}{dx^2}+4\frac{dy}{dx}+2y=0.$ (1)

The substitution of x^m for y in the first member gives

$$m(m+3)x^{m-1}-(m-2)(m+1)x^m. \tag{2}$$

This shows that, in the case of a series in ascending powers of x which is a solution of (1), the difference between the successive exponents of x is unity. (The difference between successive exponents may be denoted by s.) Hence, the required series has the form

$$A_0x^m+A_1x^{m+1}+\cdots+A_{r-1}x^{m+r-1}+A_rx^{m+r}+\cdots. \tag{3}$$

The substitution of (3) for y in the first member of (1) gives

$$\left.\begin{aligned}&A_0m(m+3)x^{m-1}+[A_1(m+1)(m+4)-A_0(m-2)(m+1)]x^m+\cdots\\&+[A_r(m+r)(m+r+3)-A_{r-1}(m+r-3)(m+r)]x^{m+r-1}+\cdots.\end{aligned}\right\} \tag{4}$$

When (3) is a solution of (1), the expression (4) is identically equal to zero, and the coefficient of each power of x therein is equal to zero. Therefore, on equating the coefficients of x^{m-1} and x^{m+r-1} to zero, it follows that

$$m=0,\ m=-3,\ A_r=\frac{m+r-3}{m+r+3}A_{r-1}. \tag{5}$$

The results (5) give the initial exponents of x and the relation between successive coefficients in the series which satisfy (1). Hence, the required series are completely determined.

For $m=0$, $A_r=\frac{r-3}{r+3}A_{r-1}$.

Then $A_1=\frac{1-3}{1+3}A_0=-\frac{1}{2}A_0,$

$A_2=\frac{2-3}{2+3}A_1=\frac{1}{10}A_0,$

$A_3=\frac{3-3}{3+3}A_2=0,$

$A_4=\frac{4-3}{4+3}A_3=0,$

$A_5=A_6=\cdots=0.$

Hence the corresponding series is

$$A_0(1-\tfrac{1}{2}x+\tfrac{1}{10}x^2).$$

For $m=-3$, $A_r=\frac{r-6}{r}A_{r-1}$.

Then $A_1=-5A_0,\ A_2=10A_0,$

$A_3=-10A_0,\ A_4=5A_0,$

$A_5=-A_0,\ A_6=A_7=\cdots=0.$

Hence the corresponding series is

$$A_0x^{-3}(1-5x+10x^2-10x^3+5x^4-x^5).$$

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* Note L contains a general discussion to be read after Exs. 1, 2.

In each of these series, A_0 is an arbitrary constant, hence a solution of (1) is

$$y = A(1 - \tfrac{1}{2}x + \tfrac{1}{10}x^2) + Bx^{-3}(1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5). \quad (6)$$

This is a general solution, since it contains two arbitrary constants.

The expression (2) also shows that, in the case of a series in descending powers of x which is a solution of (1), the difference between the successive exponents is -1. Such a series has the form

$$A_0x^m + A_1x^{m-1} + \cdots + A_{r-1}x^{m-r+1} + A_rx^{m-r} + \cdots.$$

Substitution of this in the first member of (1) gives

$$-(m-2)(m+1)A_0x^m + [A_0m(m+3) - A_1m(m-3)]\,x^{m-1} + \cdots$$
$$+ [A^{r-1}(m-r+1)(m-r+4) - A_r(m-r-2)(m-r+1)]\,x^{m-r} + \cdots.$$

On equating the coefficients of x^m, x^{m-r} to zero, it follows that

$$m = 2,\ \ m = -1,\ \ A_r = \frac{m-r+4}{m-r-2}A^{r-1}.$$

On deducing the series corresponding to these values of m in the manner shown above, it will be found that a general solution of (1) is

$$y = Ax^2(1 - 5x^{-1} + 10x^{-2} - 10x^{-3} + 5x^{-4} - x^{-5}) + Bx^{-1}(1 - \tfrac{1}{2}x^{-1} + \tfrac{1}{10}x^{-2}).$$

The first of these series is the same as the second in (6) above.

The procedure when the second member of (1) is not zero will be made clear in the first example below.

Ex. 2. Integrate (1) $x^4\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x^{-1}.$

First find the complementary function. The substitution of x^m for y in the first member gives

$$(2)\ \ m(m-1)x^{m+2} + (m+1)x^m;$$

whence $s = -2$, and $m = 0$ or 1.

The substitution of $\sum\limits_{r=0}^{r=\infty} A_rx^{m-2r}$ for y gives

$$\sum_{r=0}^{r=\infty}[(m-2r)(m-2r-1)A_rx^{m-2r+2} + (m-2r+1)A_rx^{m-2r}] = 0.$$

The coefficient of x^{m-2r+2} must vanish; therefore

$$(m-2r)(m-2r-1)A_r + (m-2r+3)A_{r-1} = 0,$$

hence
$$(3)\quad A_r = -\frac{m-2r+3}{(m-2r)(m-2r-1)}A_{r-1},$$

which is the relation between the coefficients.

For $m=0$,
$$A_r = \frac{2r-3}{2r(2r+1)}A_{r-1};$$

hence
$$A_1 = -\frac{1}{2\cdot 3}A_0,$$
$$A_2 = \frac{1}{4\cdot 5}A_1 = -\frac{1}{5\,!}A_0,$$
$$A_3 = \frac{3}{6\cdot 7}A_2 = -\frac{1\cdot 3}{7\,!}A_0, \text{ etc.};$$

the corresponding series is
$$1-\frac{1}{3\,!}x^{-2}-\frac{1}{5\,!}x^{-4}-\frac{1\cdot 3}{7\,!}x^{-6}-\frac{1\cdot 3\cdot 5}{9\,!}x^{-8}-\cdots.$$

For $m=1$,
$$A_r = \frac{2r-4}{2r(2r-1)}A_{r-1};$$

hence
$$A_1 = \frac{2-4}{2(2-1)}A_0 = -A_0,$$
$$A_2 = \frac{4-4}{4(3-1)}A_1 = 0,$$

and A_3, A_4, $\cdots$ are each equal to zero; the series in this case is finite, being
$$x-\frac{1}{x}.$$

Hence the complementary function is
$$A\left(1-\frac{1}{3\,!}x^{-2}-\frac{1}{5\,!}x^{-4}-\frac{1\cdot 3}{7\,!}x^{-6}-\cdots\right)+B\left(x-\frac{1}{x}\right).$$

In order to find the particular integral, substitute A_0x^m for y; then must $m(m-1)A_0x^{m+2}=x^{-1}$; comparison of the exponents shows that $m=-3$; and hence $A_0=\frac{1}{12}$.

For $m=-3$; the relation (3) between the coefficients becomes
$$A_r = \frac{2r}{(2r+3)(2r+4)}A_{r-1};$$

hence $A_1 = \frac{2}{5\cdot 6}A_0$, $A_2 = \frac{2\cdot 4}{5\cdot 7\cdot 6\cdot 8}A_0$, $A_3 = \frac{2\cdot 4\cdot 6}{5\cdot 7\cdot 9\cdot 6\cdot 8\cdot 10}A_0$, $\cdots$, and the particular integral is

$$\frac{x^{-3}}{12}\left(1+\frac{2}{5\cdot 6}x^{-2}+\frac{2\cdot 4}{5\cdot 7\cdot 6\cdot 8}x^{-4}+\cdots\right),$$

that is,
$$2x^{-3}\left(1+\frac{2}{6!}x^{-2}+\frac{2\cdot 4}{8\,!}x^{-4}+\cdots\right).$$

Ex. 3. Show by the method of integration in series, that the general solution of $\frac{d^2y}{dx^2}+y=0$ is $A\cos x+B\sin x$.

Ex. 4. $(2x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}+2y=0.$

Ex. 5. $2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+(1-x^2)y=x^2.$

Ex. 6. $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0.$

83. Equations of Legendre, Bessel, Riccati, and the hypergeometric series.* A fuller discussion of integration in series than is here attempted is beyond the limits of an introductory course in differential equations. The purpose of Art. 82 has merely been to give the student a little idea of a method which is of wide application; and which is used in solving four very important equations that often occur in investigations in applied mathematics, — the equations of Riccati, Bessel, Legendre, and the hypergeometric series.

Johnson's *Differential Equations*, Arts. 171–180, discusses the methods to be followed when two roots of (6) Art. 82, become equal, the corresponding series then being identical; and when two of the roots differ by a multiple of s, one series then being included in the other; and when a coefficient A_r is infinite.

The equations referred to above, and references to be consulted concerning them, are as follows:

† *Legendre's equation* is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0,$$

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* In connection with this article, the student is advised to read W. E. Byerly, *Fourier's Series and Spherical Harmonics*, Arts. 14–18.

† Adrien Marie Legendre (1752–1833) was the author of *Elements of Geometry*, published in 1794, the modern rival of Euclid. He is noted for his researches in Elliptic Functions and Theory of Numbers. He was the creator, with Laplace, of Spherical Harmonics.

or
$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y=0,$$

where n is a constant, generally a positive integer. See Ex. 5, Art. 82.

(Forsyth, *Diff. Eq.*, Arts. 89–99; Johnson, *Diff. Eq.*, Arts. 222–226; Byerly, *Fourier's Series and Spherical Harmonics*, Arts. 9, 10, 13 (c), 16, 18 (c), and Chap. V., pp. 144–194; Byerly, *Harmonic Functions* (Merriman and Woodward, *Higher Mathematics*, Chap. V.), Arts. 4, 12–17.)

* *Bessel's equation* is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0,$$

in which n is usually an integer.

(Forsyth, *Diff. Eq.*, Arts. 100–107; Johnson, *Diff. Eq.*, Arts. 215–221; Byerly, *Fourier's Series, etc.*, Arts. 11, 17, 18 (d), and Chap. VII., pp. 219–233; Byerly, *Harmonic Functions*, Arts. 5, 19–23 of Chap. V. in *Higher Mathematics;* Gray and Mathews, *Bessel Functions and their Applications to Physics;* Todhunter, *Laplace's, Lamé's, and Bessel's Functions.*)

† *Riccati's equation* is

$$\frac{dy}{dx}+by^2=cx^m,$$

to which form is reducible the equation $x\frac{dy}{dx}-ay+by^2=cx^n$. The latter equation is integrable in finite terms when $n=2a$, or when $\frac{n\pm 2a}{2n}$ is a positive integer. Riccati's equation can be reduced to a linear form, but of the second order, $\frac{d^2u}{dx^2}\pm a^2x^mu=0$.

(Forsyth, *Diff. Eq.*, Arts. 108–111; Johnson, *Diff. Eq.*, Arts. 204–214; Glaisher, *Memoir in Phil. Trans.*, 1881, pp. 759–828.)

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* Frederick Wilhelm Bessel (1784–1846) may be regarded as the founder of modern practical astronomy. In 1824, in connection with a problem in orbital motion, he introduced the functions called by his name which appear in the integrals of this equation.

† Jacopo Francesco, Count Riccati (1676–1754) is best known in connection with this equation, which was published in 1724. He integrated it for some special cases.

* *The differential equation of the hypergeometric series* is

$$\frac{d^2y}{dx^2}+\frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}\,\frac{dy}{dx}-\frac{\alpha\beta}{x(1-x)}\,y=0.$$

This equation has the hypergeometric series

$$1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma\cdot\gamma+1}x^2+\cdots$$

usually denoted by $F(\alpha, \beta, \gamma, x)$, for one of its particular integrals; and has a set of 24 particular integrals, each of which contains a hypergeometric series.

(Forsyth, *Diff. Eq.*, Arts. 113–134; Johnson, *Diff. Eq.*, Arts. 181–203.)

EXAMPLES ON CHAPTER VIII.

1. Show that the following equation is exact and find a first integral.

$$\left(y^2+2x^2\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2(y+x)\left(\frac{dy}{dx}\right)^2+x\frac{dy}{dx}+y=0.$$

2. $\dfrac{d^2y}{dx^2}-\dfrac{a^2}{x(a^2-x^2)}\,\dfrac{dy}{dx}=\dfrac{x^2}{a(a^2-x^2)}.$

3. $(1+x+x^2)\dfrac{d^3y}{dx^3}+(3+6x)\dfrac{d^2y}{dx^2}+6\dfrac{dy}{dx}=0.$

4. Find a first integral of $x^3\dfrac{d^3y}{dx^3}+4x^2\dfrac{d^2y}{dx^2}+x(x^2+2)\dfrac{dy}{dx}+3x^2y=2x.$

5. $\dfrac{d^4y}{dx^4}-a^2\dfrac{d^2y}{dx^2}=0.$

6. $\left(\dfrac{dy}{dx}\right)^2-y\dfrac{d^2y}{dx^2}=n\left\{\left(\dfrac{dy}{dx}\right)^2+a^2\left(\dfrac{d^2y}{dx^2}\right)^2\right\}^{\frac{1}{2}}.$

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* This is also called the Gaussian equation, and the series, the Gaussian series, after Karl Friedrich Gauss (1777–1855), who is regarded as one of the greatest mathematicians of the nineteenth century. He is especially noted for his invention of a new method for calculating orbits, and for his researches in the Theory of Numbers. It was Euler (see footnote, p. 64) who discovered the series and set forth its differential equation; but Gauss made important investigations concerning the series, and showed that the ordinary algebraic, trigonometrical, and exponential series can be represented by it. (For illustrations of the last remark, see Johnson, *Differential Equations*, Ex. 1, p. 220.)

7. $(x^3 - x)\dfrac{d^3y}{dx^3} + (8x^2 - 3)\dfrac{d^2y}{dx^2} + 14x\dfrac{dy}{dx} + 4y = \dfrac{2}{x^3}.$

8. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^3 = 0.$ 10. $\sin x\dfrac{d^2y}{dx^2} - \cos x\dfrac{dy}{dx} + 2y\sin x = 0.$

9. $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2.$ 11. $x\dfrac{d^3y}{dx^3} - x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0.$

12. $\dfrac{d^2y}{dx^2} = \dfrac{a}{x}.$ 13. $y(1 - \log y)\dfrac{d^2y}{dx^2} + (1 + \log y)\left(\dfrac{dy}{dx}\right)^2 = 0.$

14. $\dfrac{d^3y}{dx^3} = \sin^2 x.$ 15. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^x.$

16. $\dfrac{d^3y}{dx^3} + \cos x\dfrac{d^2y}{dx^2} - 2\sin x\dfrac{dy}{dx} - y\cos x = \sin 2x.$

17. $\sin^2 x\dfrac{d^2y}{dx^2} = 2y.$ 18. $a\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}.$ 19. $y^3\dfrac{d^2y}{dx^2} = a.$

20. Find three independent first integrals of $\dfrac{d^3y}{dx^3} = f(x).$

21. $\dfrac{d^2y}{dx^2} = a^2 + k^2\left(\dfrac{dy}{dx}\right)^2.$

Note for Art. 75. A first integral of a differential equation is an equation deduced from it of an order lower by unity than that of the original equation and containing an arbitrary constant.

First integrals of $\dfrac{d^2y}{dx^2} + y = 0$ are

$$\left(\frac{dy}{dx}\right)^2 + y^2 = A,\quad \frac{dy}{dx}\cos x + y\sin x = B,\quad -\frac{dy}{dx}\sin x + y\cos x = C,$$
$$\frac{dy}{dx} = y\cot(x + \alpha).$$

These are not all independent, for the four constants A, B, C, α, are connected by the equations

$$B = \sqrt{A}\cos\alpha,\quad C = \sqrt{A}\sin\alpha.$$

The elimination of $\dfrac{dy}{dx}$ from the second and third of these integrals gives the solution
$$y = B\sin x + C\cos x;$$
and the elimination from the first and fourth gives another form of the solution, namely,
$$y = A\sin(x + \alpha).$$

In general, if independent first integrals equal in number to the order of the equation have been obtained, all the differential coefficients can be eliminated from them so as to leave the primitive.

CHAPTER IX.

EQUATIONS OF THE SECOND ORDER.

84. There are other methods of solution, different from those shown in the last three chapters, which are applicable to some equations of the second order; Arts. 85–89 will be taken up with an exposition of three of these methods.

If a differential equation is not in a form to which any of the methods already described apply, it may be possible to put it in such a form. The very important transformations of an equation that can be effected by changing the dependent or the independent variable will be discussed in Arts. 90–92. Art. 93 will contain a synopsis of all the methods considered up to that point which may be employed in solving equations of the second order.

85. The complete solution in terms of a known integral. A theorem of great importance relating to the linear differential equation of the second order, is the following:

If an integral included in the complementary function of such an equation be known, the complete solution can be expressed in terms of the known integral.

Suppose that $y = y_1$ is a known integral in the complementary function of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X; \qquad (1)$$

then the complete solution of (1) can be determined in terms of y_1.

Let $$y = y_1 v$$

be another solution of (1); v will now be determined. On substituting $y_1 v$ for y in (1), it will become

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}=\frac{X}{y_1}; \tag{2}$$

since, by hypothesis,

$$\frac{d^2y_1}{dx_2}+P_1\frac{dy_1}{dx}+Qy_1=0.$$

On putting p for $\frac{dv}{dx}$, (2) becomes

$$\frac{dp}{dx}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)p=\frac{X}{y_1}; \tag{3}$$

and this equation, being linear and of the first order, can be solved for p. On using the method of Art. 20, the solution is found to be

$$p=\frac{dv}{dx}=\frac{c_1e^{-\int Pdx}}{y_1^2}+\frac{e^{-\int Pdx}}{y_1^2}\int y_1e^{\int Pdx}X\,dx;$$

whence, integrating,

$$v=c_2+c_1\int\frac{e^{-\int Pdx}}{y_1^2}\,dx+\int\frac{e^{-\int Pdx}}{y_1^2}\int y_1e^{\int Pdx}X\,(dx)^2.$$

Therefore another solution of (1) is

$$y=y_1v=c_2y_1+c_1y_1\int\frac{e^{-\int Pdx}}{y_1^2}\,dx+y_1\int\frac{e^{-\int Pdx}}{y_1^2}\int y_1e^{\int Pdx}X\,(dx)^2. \tag{4}$$

This includes the given solution $y=y_1$; and, since it contains two arbitrary constants, it is the complete solution. From the form of the solution (4), it is evident that the second part of the complementary function is $y_1\int\frac{e^{-\int Pdx}}{y_1^2}\,dx$, and that the particular integral is $y_1\int\frac{e^{-\int Pdx}}{y_1^2}\int y_1e^{\int Pdx}X\,(dx)^2$.

86. Relation between the integrals. It is easily shown, that if $y = y_1$, $y = y_2$ be two independent integrals of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0,$$

then
$$y_1\frac{dy_2}{dx} - y_2\frac{dy_1}{dx} = ce^{-\int P dx}.$$

(See Forsyth's *Diff. Eq.*, Art. 65; Johnson's *Diff. Eq.*, Art. 147.)

It may also be remarked in passing, that the deduction of (3) Art. 85 from (1), when an integral of the latter is known, is an example of the theorem: that, if one or several independent integrals of a linear equation be known, the order of the equation can be lowered by a number equal to the number of the known integrals.

(See Forsyth's *Diff. Eq.*, Arts. 41, 76, 77.)

87. To find the solution by inspection. Since the complete integral of (1) Art. 85 can be found if one integral in its complementary function be known, it is generally worth while to try whether an integral in the latter can be determined by inspection.

Ex. 1. Solve $x\frac{d^2y}{dx^2} + (1 - x)\frac{dy}{dx} - y = e^x$.

Here, the sum of the coefficients being zero, e^x is obviously a solution of

$$x\frac{d^2y}{dx^2} + (1 - x)\frac{dy}{dx} - y = 0.$$

Substitution of ve^x for y in the original equation gives

$$x\frac{d^2v}{dx^2} + (1 + x)\frac{dv}{dx} = 1;$$

this, on substituting p for $\frac{dv}{dx}$, becomes

$$x\frac{dp}{dx} + (1 + x)p = 1,$$

a linear equation of the first order. Its solution is

$$p = \frac{dv}{dx} = c_1\frac{e^{-x}}{x} + \frac{1}{x};$$

hence $$v = \log x + c_1 \int x^{-1} e^{-x} dx + c_2;$$

and therefore the complete solution is

$$y = e^x \log x + c_1 e^x \int x^{-1} e^{-x} dx + c_2 e^x.$$

Equation (4) Art. 85 might have been used as a substitution formula, but it is better to work out each example by the same general method by which (4) was itself derived.

Ex. 2. Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = x.$

[Here, $y = x$ is obviously a solution when the second member is zero. A solution can often be found by an inspection of the terms of lower order in the equation.]

Ex. 3. Solve $(3 - x)\frac{d^2y}{dx^2} - (9 - 4x)\frac{dy}{dx} + (6 - 3x)y = 0.$

Ex. 4. Solve $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$, given that $x + \frac{1}{x}$ is one integral.

88. The solution found by means of operational factors. Suppose that the linear equation of the second order

$$P_0\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = X,$$

is expressed in the form $f(D)y = X.$

Sometimes $f(D)$ can be resolved into a product of two factors $F_1(D)$ and $F_2(D)$, such that, when $F_1(D)$ operates upon y, and then $F_2(D)$ operates upon the result of this operation, the same result is obtained as when $F(D)$ operates upon y. This may be expressed symbolically,

$$f(D)y = F_2(D)\{F_1(D)y\};$$

or simply, $$f(D)y = F_2(D)F_1(D)y,$$

it being understood that the operations indicated in the second member of the last equation are made in order from right to left. Factors of this kind have already been employed in dealing with linear equations with constant coefficients, and with the homogeneous linear equations, Arts. 53, 55, 67, etc. With the exception of the classes of equations just mentioned, the factors are generally not commutative; this can be verified in the case of the examples below.

If one of the integrals be known, its corresponding factor is known, and the second factor can be determined by means of the equation and the known factor. For instance, if $y = e^x$ be an integral of the given equation, then $(D-1)y$ is the corresponding factor; if $y = x$ be an integral, $(xD-1)y$ is the corresponding factor. The following example will make the method of procedure clear.

Ex. 1. Solve $x\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} - y = e^x$.

This equation, which is Ex. 1, Art. 87, when written in the symbolic form, is

$$[xD^2 + (1-x)D - 1]y = e^x; \tag{1}$$

on using symbolic factors, it becomes

$$(xD+1)(D-1)y = e^x. \tag{2}$$

[These factors are not commutative, for $(D-1)(xD+1)y$ on expansion gives $\{xD^2 + (2-x)D - 1\}y$].

Let $$(D-1)y = v, \tag{3}$$

and (2) becomes $(xD+1)v = e^x$; whence, $v = cx^{-1} + e^x x^{-1}$.

Substitution of this value of v in (3) gives

$$(D-1)y = cx^{-1} + e^x x^{-1};$$

whence, on integrating, $y = c_1 e^x + ce^x \int e^{-x}x^{-1}dx + e^x \log x$,

the solution found in the last article.

Ex. 2. Solve Ex. 3, Art. 87, by this method.

Ex. 3. Solve $3x^2\dfrac{d^2y}{dx^2} + (2-6x^2)\dfrac{dy}{dx} - 4y = 0$.

Ex. 4. Solve $3x^2\dfrac{d^2y}{dx^2} + (2+6x-6x^2)\dfrac{dy}{dx} - 4y = 0$.

89. Solution found by means of two first integrals. It follows, from a statement made in Art. 75, that a linear equation of the second order has two first integrals of the first order. If these integrals be known, then $\frac{dy}{dx}$ can be eliminated between them; the relation thus found between x and y will be a solution of the original equation.

Another method of solution that can be used in the case of the linear equation of the second order is the "method of variation of parameters."* As most of the equations solvable by it are solvable in other ways, and as it is rather long, it will not be given here. (See Johnson's *Diff. Eq.*, Arts. 90, 91; Forsyth's *Diff. Eq.*, Arts. 65–67.)

Ex. Solve $a^2\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$ by means of the first integrals.

On putting $\frac{dy}{dx} = p$, $\frac{d^2y}{dx^2} = \frac{dp}{dx}$, and integrating, there appears a first integral

$$p + \sqrt{1 + p^2} = e^{\frac{x+c_1}{a}}.$$

On substituting for $\frac{d^2y}{dx^2}$ its equivalent expression $p\frac{dp}{dy}$, and integrating, another first integral is obtained,

$$a^2p^2 = (y + c_2)^2 - a^2.$$

The elimination of p between these first integrals gives the solution

$$y + c_2 = a \cosh \frac{x + c_1}{a}.$$

90. Transformation of the equation by changing the dependent variable. Sometimes an equation can be transformed into an integrable type by changing the dependent variable. If any linear equation of the second order,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X, \tag{1}$$

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* This method is due to Lagrange.

be taken, and $y_1 v$ be substituted for y therein, y_1 being some function of x, (1) will be transformed into

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{y_1}\frac{dy_1}{dx}\right)\frac{dv}{dx}+\frac{1}{y_1}\left(\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1\right)v=\frac{X}{y_1}, \qquad (2)$$

which has v for its dependent variable.

This equation may be written

$$\frac{d^2v}{dx^2}+P_1\frac{dv}{dx}+Q_1v=\frac{X}{y_1}, \qquad (3)$$

where $$P_1=P+\frac{2}{y_1}\frac{dy_1}{dx}, \qquad (4)$$

and $$Q_1=\frac{1}{y_1}\left(\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1\right). \qquad (5)$$

Any value desired can be assigned to P_1 *or* Q_1 by means of a proper choice of y_1. Thus, Q_1 will be zero if y_1 be chosen so that

$$\frac{d^2y_1}{dx^2}+P\frac{dy_1}{dx}+Qy_1=0;$$

this is what was done in Art. 85.

Again, P_1, the coefficient of the first derivative in (3), can have any arbitrary value assigned to it; but then y_1 must be chosen so as to satisfy (4); that is,

$$y_1=e^{\frac{1}{2}\int(P_1-P)dx}. \qquad (6)$$

91. Removal of the first derivative. In particular, it follows from (4) or (6) Art. 90 that P_1 is zero

if $$y_1=e^{-\frac{1}{2}\int Pdx}.$$

On substituting this value of y_1 in the coefficient of v in (2) Art. 90, this coefficient becomes

$$Q-\tfrac{1}{2}\frac{dP}{dx}-\tfrac{1}{4}P^2.$$

Therefore the differential equation (1) Art. 90 of the second order is transformed into a differential equation not containing the first derivative, by substituting

$$ve^{-\frac{1}{2}\int P dx} \text{ for } y;$$

and the transformed equation is

$$\frac{d^2v}{dx^2} + Q_1 v = X_1, \tag{1}$$

where $Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$, and $X_1 = Xe^{\frac{1}{2}\int P dx}$.

The new equation (1) may happen to be easily integrable. Transforming (1) Art. 90 into the form (1) is called "removing the first derivative." The student should memorise the above values of the new Q_1 and X_1, in terms of P, Q, X, for then he can immediately write down the new equation in v, without the labour of making the substitution in the original equation and reducing.

It may be remarked in passing that this removal of the term next to the second derivative is merely an example of the general theorem, that the coefficient of the term of $(n-1)$th order in a linear equation

$$\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n y = X,$$

can be removed by substituting $y_1 v$ for y, where

$$y_1 = e^{-\frac{1}{n}\int P_1 dx}.$$

The reader can easily verify this by making the substitution. (See Forsyth's *Diff. Eq.*, Art. 42.)

Ex. 1. Solve $\frac{d^2y}{dx^2} + \frac{1}{x^{\frac{1}{3}}}\frac{dy}{dx} + \left(\frac{1}{4x^{\frac{2}{3}}} - \frac{1}{6x^{\frac{4}{3}}} - \frac{6}{x^2}\right)y = 0.$

Here $P = x^{-\frac{1}{3}}$, $Q = \frac{1}{4x^{\frac{2}{3}}} - \frac{1}{6x^{\frac{4}{3}}} - \frac{6}{x^2}$; and hence $y_1 = e^{-\frac{1}{2}\int P dx} = e^{-\frac{3}{4}x^{\frac{2}{3}}}$.

If the second term be removed,

$$Q_1 = Q - \tfrac{1}{2}\frac{dP}{dx} - \tfrac{1}{4}P^2 = -\frac{6}{x^2};$$

and hence the transformed equation is

$$\frac{d^2v}{dx^2} - \frac{6v}{x^2} = 0,$$

the solution of which is

$$v = c_1x^3 + \frac{c_2}{x^2}.$$

Hence the general solution of the given equation is

$$y = y_1v = e^{-\frac{3}{4}x^{\frac{4}{3}}}\left(c_1x^3 + \frac{c_2}{x^2}\right).$$

Ex. 2. Solve $4x^2\dfrac{d^2y}{dx^2} + 4x^5\dfrac{dy}{dx} + (x^8 + 6x^4 + 4)y = 0.$

Ex. 3. Solve $\dfrac{d^2y}{dx^2} - 2\tan x\dfrac{dy}{dx} + 5y = 0.$

Ex. 4. Solve $x^2\dfrac{d^2y}{dx^2} - 2(x^2 + x)\dfrac{dy}{dx} + (x^2 + 2x + 2)y = 0.$

92. Transformation of the equation by changing the independent variable. An equation can sometimes be transformed into an integrable form by changing the independent variable.

Suppose that
$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \tag{1}$$

is any linear equation of the second order, and that the independent variable is to be changed from x to z, there being some given relation, $z = f(x)$, connecting x and z.

Since $\dfrac{dy}{dx} = \dfrac{dy}{dz}\dfrac{dz}{dx}$, and $\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dz^2}\left(\dfrac{dz}{dx}\right)^2 + \dfrac{dy}{dz}\dfrac{d^2z}{dx^2}$,

(1) becomes
$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = X_1, \tag{2}$$

where
$$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2},\quad Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2},\ \text{and}\ X_1 = \frac{X}{\left(\dfrac{dz}{dx}\right)^2}. \tag{3}$$

P_1, Q_1, X_1, as just expressed, are functions of x; but can be immediately expressed as functions of z by means of the relation connecting z and x.

Any arbitrary value can be given to P_1; but then z must be so chosen that it satisfies the first of equations (3). In particular, P_1 will be zero if

$$\frac{d^2z}{dx^2}+P\frac{dz}{dx}=0, \text{ that is, if } z=\int e^{-\int Pdx}dx.$$

Again, the new coefficient Q_1 will be a constant, a^2, by virtue of the second of equations (3), if

$$a^2\left(\frac{dz}{dx}\right)^2=Q, \text{ that is, if } az=\int\sqrt{Q}\,dx.$$

Ex. 1. Solve $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}+\dfrac{a^2}{x^4}y=0.$

Find z, such that $\left(\dfrac{dz}{dx}\right)^2=\dfrac{a^2}{x^4}$; solving, $z=\pm\dfrac{a}{x}$.

Change of the independent variable from x to z will now give

$$\frac{d^2y}{dz^2}+y=0;$$

and this has for its solution

$$y=A\cos z+B\sin z.$$

Hence the solution of the given equation is

$$y=c_1\cos\frac{a}{x}+c_2\sin\frac{a}{x}.$$

Ex. 2. Solve $\dfrac{d^2y}{dx^2}+\cot x\dfrac{dy}{dx}+4\,y\operatorname{cosec}^2x=0.$

Ex. 3. Solve $x\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+4\,x^3y=x^5.$

Ex. 4. Solve $x^6\dfrac{d^2y}{dx^2}+3\,x^5\dfrac{dy}{dx}+a^2y=\dfrac{1}{x^2}.$

93. Synopsis of methods of solving equations of the second order. This article is merely a synopsis of all the methods discussed thus far in the book that are employed in the solution of equations of the second order. Several of these methods may be suitable for solving the same equation. The references are to the chapters and articles where the methods are described. The student is advised to select a few equations of the second order from the articles referred to, and to solve each one in two or more different ways.

An equation of the second order may be

(*a*) linear with constant coefficients, [Chap. VI.];
(*b*) a homogeneous linear equation, [Chap. VII.];
(*c*) an exact differential equation, [Arts. 73–75, 76];
(*d*) an equation that does not directly contain the dependent variable, [Arts. 76, 78];
(*e*) an equation that does not directly contain the independent variable, [Arts. 77, 79];
(*f*) in the form $\frac{d^2y}{dx^2}=f(y)$, [Art. 77];
(*g*) an equation, one of whose integrals is known or is easily found by inspection, [Arts. 85, 87];
(*h*) factorable into symbolic operators, [Art. 88];
(*i*) an equation of which two first integrals can be easily found, [Art. 89];
(*j*) an equation that can be integrated in series. [Art. 82].

If the equation is not in an integrable form, it may be put in such a form by

(*a*) so changing the dependent variable, that (1) the coefficient of the first derivative will have an assigned value [Art. 90];
or that (2) (in particular), this coefficient will be zero [Art. 91];
(*b*) so changing the independent variable, that (1) the equation will be transformed into the linear form with constant coefficients, or into the homogeneous linear form [Art. 71];

or that (2) the coefficient of the first derivative will have an assigned value, and, in particular, the value zero [Art. 92];

or that (3) the coefficient of the variable will have an assigned value, and, in particular, be a constant [Art. 92].

EXAMPLES ON CHAPTER IX.

1. $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}=n^2y.$ 3. $\dfrac{d^2y}{dx^2}-\dfrac{2}{x}\dfrac{dy}{dx}+\left(n^2+\dfrac{2}{x^2}\right)y=0.$

2. $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}+n^2y=0.$ 4. $(1+x^2)\dfrac{d^2y}{dx^2}+3x\dfrac{dy}{dx}+y=0.$

5. $(x-3)\dfrac{d^2y}{dx^2}-(4x-9)\dfrac{dy}{dx}+3(x-2)y=0$, e^x being a solution.

6. $\dfrac{d^2y}{dx^2}-2bx\dfrac{dy}{dx}+b^2x^2y=0.$

7. $\dfrac{d^2y}{dx^2}+4x\dfrac{dy}{dx}+4x^2y=0.$

8. $x\dfrac{d^2y}{dx^2}-(2x-1)\dfrac{dy}{dx}+(x-1)y=0$, given that $y=e^x$ is a solution.

9. $(1-x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x(1-x^2)^{\frac{3}{2}}.$

10. $(x\sin x+\cos x)\dfrac{d^2y}{dx^2}-x\cos x\dfrac{dy}{dx}+y\cos x=0$, of which $y=x$ is a solution.

11. $x^2\dfrac{d^3y}{dx^3}+x^2\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0.$

12. $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-a^2y=0$, of which $y=ce^{a\sin^{-1}x}$ is an integral.

13. $\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=f(x).$

14. $x^2\dfrac{d^2y}{dx^2}-2x(1+x)\dfrac{dy}{dx}+2(1+x)y=x^3.$

15. $(a^2-x^2)\dfrac{d^2y}{dx^2}-\dfrac{a^2}{x}\dfrac{dy}{dx}+\dfrac{x^2}{a}y=0.$

16. $(x^3-x)\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+n^2x^3y=0.$ 18. $y\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2+1=0.$

17. $x^2y\dfrac{d^2y}{dx^2}+\left(x\dfrac{dy}{dx}-y\right)^2=0.$ 19. $x^4\dfrac{d^2y}{dx^2}+2x^3\dfrac{dy}{dx}+n^2y=0.$

CHAPTER X.

GEOMETRICAL AND PHYSICAL APPLICATIONS.

94. Chapter V. was devoted to geometrical and physical applications; but the choice of problems for that chapter was restricted by the condition that a differential equation of an order higher than the first should not be needed in determining their solution. The practical problems now to be given are of the same general character as those already set; but in order to obtain their solution, equations of orders higher than the first may be required.

95. Geometrical Problems. The following can be added to the geometrical principles and formulæ given in Art. 42.

The radius of curvature in rectangular co-ordinates is

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

If the normal be always drawn towards the x-axis, both it and the radius of curvature at any point on the curve are drawn in the same direction when y and $\frac{d^2y}{dx^2}$ at the point are opposite in sign, and they are drawn in opposite directions when y and $\frac{d^2y}{dx^2}$ agree in sign. This will be apparent on drawing four curves, one concave upward and one concave downward, above the x-axis, and two similar ones below this axis.

Ex. Find the equation of the curve for any point of which the second derivative of the ordinate is inversely proportional to the semi-cubical power of the product of the sum and difference of the abscissa and a constant length a; determine the curve so that it will cut the y-axis at right angles, and the x-axis at a distance a from the origin.

The first condition is expressed by either of the equations

$$\frac{d^2y}{dx^2} = \frac{k^2}{(x^2 - a^2)^{\frac{3}{2}}}, \text{ and } \frac{d^2y}{dx^2} = \frac{k^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

Integrating the first equation,

$$\frac{dy}{dx} = -\frac{k^2}{a^2}\frac{x}{\sqrt{x^2 - a^2}} + c_1;$$

but, by the second condition, $\frac{dy}{dx} = 0$ when $x = 0$, and hence $c_1 = 0$; this gives

$$\frac{dy}{dx} = -\frac{k^2}{a^2}\frac{x}{\sqrt{x^2 - a^2}}.$$

Integrating, $$y = -\frac{k^2}{a^2}\sqrt{x^2 - a^2} + c;$$

but, by the third condition, $y = 0$ when $x = a$, and hence $c = 0$; therefore the equation of the curve reduces to

$$k^4x^2 - a^4y^2 = a^2k^4,$$

the equation of an hyperbola with transverse axis equal to $2a$, and conjugate axis equal to $\frac{2k^2}{a}$.

Had the second equation been taken, the equation of an ellipse, $k^4x^2 + a^4y^2 = a^2k^4$, would have been obtained.

96. Mechanical and physical problems. The following can be added to the mechanical principles and formulæ given in Art. 48; s, v, x, y, r, θ, t, have the same signification as before.

$\frac{d^2s}{dt^2}$ = the acceleration of the moving particle, at any point in its path.

$\frac{d^2x}{dt^2}$ = the component of the acceleration parallel to the x-axis.

$\frac{d^2y}{dt^2}$ = the component of the acceleration parallel to the y-axis.

$$\frac{d^2s}{dt^2}=\left[\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2\right]^{\frac{1}{2}}.$$

$\frac{d^2\theta}{dt^2}$ = the angular acceleration about a fixed point.

The force acting upon a particle is equal to the product of the mass of the particle by the acceleration of the motion of the particle due to the force.*

An attracting force causes negative acceleration, and a repelling force causes positive acceleration, if the centre of force be taken as origin.

Ex. **1.** Find the distance passed over by a moving point when its acceleration is directly proportional to its distance from a fixed point, the acceleration being directed towards the point from which distance is measured.

Here $$\frac{d^2s}{dt^2}=-k^2s.$$

Using the method of Art. 78,

$$2\frac{ds}{dt}\cdot\frac{d^2s}{dt^2}=-2k^2s\frac{ds}{dt},$$

whence $$\left(\frac{ds}{dt}\right)^2=k^2(a^2-s^2),$$

where k^2a^2 conveniently represents the constant of integration.

Hence $$\frac{ds}{\sqrt{a^2-s^2}}=kdt;$$

integrating, $$\sin^{-1}\frac{s}{a}=kt+b;$$

hence $$s=a\sin(kt+b).$$

Also, s can be found directly, without finding $\frac{ds}{dt}$, by the method in Chap. VI. The equation may be written

$$(D^2+k^2)s=0;$$

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* A particular choice of units is presupposed in this statement.

hence $$s = c_1 \sin kt + c_2 \cos kt;$$

that is, $$s = a \sin(kt + b),$$

as above.

Ex. 2. In the case of the simple pendulum of length l, the equation connecting the acceleration due to gravity and the angle θ through which the pendulum swings is

$$l\frac{d^2\theta}{dt^2} + g\theta = 0,$$

when θ is small. Determine the time of an oscillation.

Since $$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0,$$

$$\theta = c_1 \cos\sqrt{\frac{g}{l}}\,t + c_2 \sin\sqrt{\frac{g}{l}}\,t.$$

Let $\theta = \theta_0$ and $\frac{d\theta}{dt} = 0$ when $t = 0$; applying these conditions, $c_1 = \theta_0$, $c_2 = 0$, and hence

$$\theta = \theta_0 \cos\sqrt{\frac{g}{l}}\,t; \text{ that is, } t = \sqrt{\frac{l}{g}}\cos^{-1}\frac{\theta}{\theta_0},$$

which is the time of swing from θ_0 to θ. If $\theta = -\theta_0$, $t = \pi\sqrt{\frac{l}{g}}$; hence the time of a complete oscillation from θ_0 to $-\theta_0$ and back again is $2\pi\sqrt{\frac{l}{g}}$.

EXAMPLES ON CHAPTER X.

1. Determine the curve in which the curvature is constant and equal to k.

2. Determine the curve whose radius of curvature is equal to the normal and in the opposite direction.

3. Determine the curve whose radius of curvature is equal to the normal and in the same direction.

4. Determine the curve whose radius of curvature is equal to twice the normal and in the opposite direction.

5. Find the curve whose radius of curvature is double the normal and in the same direction.

6. Determine the curve whose radius of curvature varies as the cube of the normal.

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7. Find the curve whose radius of curvature varies inversely as the abscissa.

8. Find the distance passed over by a moving particle when its acceleration is directly proportional to its distance from a fixed point, the acceleration being directed away from the point from which distance is measured.

9. Find the distance passed over by a particle whose acceleration is constant and equal to a, v_0 being the initial velocity, and s_0 the initial distance of the particle from the point whence distance is measured.

10. Find the distance passed over by a particle when the acceleration is inversely proportional to the square of the distance from a fixed point.

11. Find the distance passed over by a body falling from rest, assuming that the resistance of the air is proportional to the square of the velocity.

12. The acceleration of a moving particle being proportional to the cube of the velocity and negative, find the distance passed over in time t, the initial velocity being v_0, and the distance being measured from the position of the particle at the time $t = 0$.

13. The relation between the small horizontal deflection θ of a bar magnet under the action of the earth's magnetic field is

$$A\frac{d^2\theta}{dt^2} + MH\theta = 0,$$

where A is the moment of inertia of the magnet about the axis, M the magnetic moment of the magnet, and H the horizontal component of the intensity of the field due to the earth. Find the time of a complete vibration.

14. In the case of a stretched elastic string, which has one end fixed and a particle of mass m attached to the other end, the equation of motion is

$$m\frac{d^2s}{dt^2} = -\frac{mg}{e}(s - l),$$

where l is the natural length of the string, and e its elongation due to a weight mg. Find s and v, determining the constants so that $s = s_0$ at the time $t = 0$, and $v = 0$ when $t = 0$.

15. A particle moves in a straight line under the action of an attraction varying inversely as the $(\frac{3}{2})$th power of the distance. Show that the velocity acquired by falling from an infinite distance to a distance a from the centre is equal to the velocity which would be acquired in moving from rest at a distance a to a distance $\frac{a}{4}$.

16. A particle moves in a straight line from rest at a distance a towards a centre of attraction, the attraction varying inversely as the cube of the distance. Find the whole time of motion.

17. The differential equation for a circuit containing resistance, self-induction, and capacity, in terms of the current and the time, is

$$\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=\frac{1}{L}f'(t),$$

$f(t)$ being the electromotive force. Find the current i.

18. The differential equation for the above circuit in terms of the charge of electricity in the condenser is

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=\frac{1}{L}f(t).$$

Find the charge q.

19. Solve $\frac{d^2i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{i}{LC}=0$ when $R^2C=4L$.

20. Solve $L\frac{di}{dt}+\frac{\int idt}{C}=0$, the differential equation which means that the self-induction and capacity in a circuit neutralize each other. Determine the constants in such a way that I is the maximum current, and $i=0$ when $t=0$.

(The given equation, on differentiation, reduces to $\frac{d^2i}{dt^2}+\frac{i}{LC}=0$.)

21. When the galvanometer is damped, the equation of motion may be written

$$\frac{d^2\theta}{dt^2}+2k\frac{d\theta}{dt}+\omega^2(\theta-\alpha)=0,$$

α being the deflection of the needle from the position from which angles are measured, when in its position of equilibrium, the factor k depending on the damping, and ω^2 on the restoring couple. Find the position of the needle at any instant.

(Emtage, *Electricity and Magnetism*, pp. 179, 180.)

* **22.** Find the equation of the elastic curve for a cantilever beam of uniform cross-section and length l, with a load P at the free end, the differential equation being

$$EI\frac{d^2y}{dx^2}=-Px,$$

where I is the moment of inertia of the cross-section with respect to the neutral axis, and E is the coefficient of elasticity of the material of the beam. (The origin being taken at the free end of the beam, the x-axis being along its horizontal projection, and the y-axis being the vertical, $\frac{dy}{dx} = 0$ when $x = l$, and $y = 0$ when $x = 0$. These conditions are sufficient to determine the constants.)

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* Merriman, *Mechanics of Materials*, pp. 72, 73.

* **23**. Find the elastic curve when the load is uniformly distributed over the beam described in Ex. 22, say w per linear unit, the differential equation being

$$EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}.$$

† **24**. Find the elastic curve for the beam considered in Ex. 23, when a horizontal tensile force Q is applied at the free end, the differential equation being

$$EI\frac{d^2y}{dx^2} = Qy - \tfrac{1}{2}wx^2.$$

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* Merriman, *Mechanics of Materials*, pp. 72, 73.

† Merriman and Woodward, *Higher Mathematics*, Prob. 105, p. 153.

CHAPTER XI.

ORDINARY DIFFERENTIAL EQUATIONS WITH MORE THAN TWO VARIABLES.

97. So far equations containing two variables have been considered. It is now necessary to treat a few forms containing more than two variables. Such equations are either *ordinary* or *partial*, the former having only one independent variable, and the latter more than one. In this chapter ordinary differential equations will be discussed.

98. Simultaneous differential equations which are linear. First will be considered the case in which there is a set of relations consisting of as many simultaneous equations as there are dependent variables; moreover, all the equations are to be linear.

By following a method somewhat analogous to that employed in solving sets of simultaneous algebraic equations that involve several unknowns, the dependent variables corresponding to the unknowns, there is obtained, by a process of elimination, an equation that involves only one dependent variable with the independent variable; and from this newly formed equation an integral relation between these two variables may be derived. Then a relation between a second dependent variable and the independent variable can be deduced, either (1) by the method of elimination and integration employed in the case of the first variable; or (2) by substituting the value already found for the first variable, in one of the equations involving only the first and second dependent variables and the independent variable. The complete solution consists of as many independent relations between the variables as there are dependent variables.

The following example will make the process clear:

Ex. 1. Solve the simultaneous equations,

$$(1)\quad \frac{dx}{dt}+\frac{dy}{dt}+2x+y=0.$$

$$(2)\quad \frac{dy}{dt}+5x+3y=0.$$

Differentiation of (2) gives

$$(3)\quad \frac{d^2y}{dt^2}+5\frac{dx}{dt}+3\frac{dy}{dt}=0.$$

These three equations suffice for the elimination of x and $\frac{dx}{dt}$; this elimination is effected by multiplying the first equation by -5, the second by 2, the third by 1, and adding; the result is

$$(4)\quad \frac{d^2y}{d^2t}+y=0.$$

Solving (4),

$$y=A\cos t+B\sin t;$$

and substituting this value of y in (2),

$$x=-\frac{3A+B}{5}\cos t+\frac{A-3B}{5}\sin t.$$

By using the symbol D, which was employed in Chap. VI., the elimination can be effected more easily. On substituting D for $\frac{d}{dt}$, the given equations become

$$(D+2)x+(D+1)y=0,$$
$$5x+(D+3)y=0.$$

Eliminating x as if D were an algebraic multiplier,

$$(D^2+1)y=0,$$

which is equation (4); the remainder of the work is as above.

If y had been eliminated instead of x, the resulting equation would have been

$$(D^2+1)x=0;$$

whence

$$x=A'\cos t+B'\sin t;$$

substitution of this value in

$$(5)\quad \frac{dx}{dt} - 3x - 2y = 0,$$

which is (1) minus (2), gives

$$y = -\frac{3B' + A'}{2}\sin t + \frac{B' - 3A'}{2}\cos t.$$

Substitution is made in (5), because it is easier to derive the value of y from it than from (1) or (2).

The second form of solution comes from the first on substituting A' for $-\frac{3A+B}{5}$, and B' for $\frac{A-3B}{5}$, the coefficients in the first value of x. In general the constants are arbitrary in the value of only one of the dependent variables.

Ex. 2. Solve $\left.\begin{aligned}\frac{dx}{dt} - 7x + y &= 0\\ \frac{dy}{dt} - 2x - 5y &= 0\end{aligned}\right\}$

Ex. 3. Solve $\left.\begin{aligned}\frac{dx}{dt} + 2x - 3y &= t\\ \frac{dy}{dt} - 3x + 2y &= e^{2t}\end{aligned}\right\}$

Ex. 4. Solve $\left.\begin{aligned}4\frac{dx}{dt} + 9\frac{dy}{dt} + 44x + 49y &= t\\ 3\frac{dx}{dt} + 7\frac{dy}{dt} + 34x + 38y &= e^{t}\end{aligned}\right\}$

Ex. 5. Solve $\left.\begin{aligned}\frac{d^2x}{dt^2} - 3x - 4y &= 0\\ \frac{d^2y}{dt^2} + x + y &= 0\end{aligned}\right\}$

99. Simultaneous equations of the first order. Simultaneous equations of the first order and of the first degree in the derivatives can sometimes be solved by the following method, which is generally shorter than that shown in the last article. Equations involving only three variables will be considered; the method, however, is general, and can be applied to equations having any number of variables.

The general type of a set of simultaneous equations of the first order between three variables is

$$\left.\begin{aligned} P_1dx + Q_1dy + R_1dz = 0 \\ P_2dx + Q_2dy + R_2dz = 0 \end{aligned}\right\}, \tag{1}$$

where the coefficients are functions of x, y, z.

These equations can be expressed in the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \tag{2}$$

where P, Q, R, are functions of x, y, z; for, on taking z as the independent variable and solving equations (1) for $\frac{dx}{dz}$ and $\frac{dy}{dz}$,

$$\frac{dx}{dz} = \frac{Q_1R_2 - Q_2R_1}{P_1Q_2 - P_2Q_1}, \quad \frac{dy}{dz} = \frac{R_1P_2 - P_1R_2}{P_1Q_2 - P_2Q_1},$$

whence $$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1},$$

which is the form (2) above.

In what follows, equations (2) will be taken as the type of a set of simultaneous equations of the first order.

If one of the variables be absent from two members of (2), the method of procedure is obvious. For example, suppose that z is absent from P and Q; then the solution of

$$\frac{dx}{P} = \frac{dy}{Q}$$

gives a relation between x and y, which is one equation of the complete solution. This equation may enable us to eliminate x or y from one of the other equations in (2), and then another integral relation may be found; this will be the second equation of the solution.

Since, by a well-known principle of algebra, the equal fractions $\frac{dx}{P}$, $\frac{dy}{Q}$, $\frac{dz}{R}$, are also equal to

$$\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}, \quad \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}, \quad \text{etc.},$$

the system of equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}, \tag{3}$$

are all satisfied by the same relations between x, y, z, that satisfy (2).

It may be possible by a proper choice of multipliers, l, m, n, l', m', n', etc., to obtain equations which are easily solved, and whose solutions are the solutions of (2). In particular, l, m, n, may be found such that

$$lP + mQ + nR = 0, \tag{4}$$

and consequently

$$l\,dx + m\,dy + n\,dz = 0. \tag{5}$$

If $l\,dx + m\,dy + n\,dz$ be an exact differential, say du, then

$$u = a$$

is one equation of the complete solution.

If l', m', n', can be chosen so that $l'P + m'Q + n'R = 0$, and $l'dx + m'dy + n'dz$ is at the same time an exact differential, dv, then, since dv is also equal to zero,

$$v = b$$

is the second equation of the complete solution. The two component solutions must be independent.

Ex. 1. Solve the simultaneous equations $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2\,xy} = \dfrac{dz}{2\,xz}$.

The equation formed by the last two fractions reduces to

$$\frac{dy}{y} = \frac{dz}{z},$$

which has for its solution

$$y = az.$$

Using x, y, z, as multipliers like l, m, n, above,

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2\,xy} = \frac{dz}{2\,xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}.$$

The equation formed by the last two fractions has for its solution

$$x^2 + y^2 + z^2 = bz.$$

The complete solution consists of these two independent solutions.

Ex. **2**. Solve $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dz}{ly - mx}$.

By using the multipliers l, m, n, one gets the equal fraction

$$\frac{l\,dx + m\,dy + n\,dz}{0};$$

therefore $$l\,dx + m\,dy + n\,dz = 0;$$

whence $$lx + my + nz = c_1.$$

The multipliers x, y, z, used in a similar manner, give

$$x\,dx + y\,dy + z\,dz = 0,$$

whence $$x^2 + y^2 + z^2 = k^2.$$

These two integrals form the complete integral of the set of equations.

Ex. **3**. Solve $\dfrac{x\,dx}{y^2z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$.

Ex. **4**. Solve $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{nxy}$.

Ex. **5**. Solve $\dfrac{dx}{y^2} = \dfrac{dy}{x^2} = \dfrac{dz}{x^2y^2z^2}$.

Ex. **6**. Solve $\dfrac{a\,dx}{(b - c)yz} = \dfrac{b\,dy}{(c - a)zx} = \dfrac{c\,dz}{(a - b)xy}$.

100. The general expression for the integrals of simultaneous equations of the first order. If the first member,

$$l\,dx + m\,dy + n\,dz,$$

of (5) Art. 99 be an exact differential, du, then, since

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz,$$

l, m, n, are proportional to $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial u}{\partial z}$, respectively; and therefore, (4) Art. 99 may be written

$$P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z}=0. \tag{1}$$

Hence, if $u=a$ be one of the integral equations of the system (2) Art. 99, then $u=a$ also satisfies (1).

Conversely, if $u=a$ be an integral of (1), it is also an integral of the system (2) Art. 99. For, since the denominator of

$$\frac{\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz}{P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z}},$$

which is formed by means of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ and the multipliers $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$, is thus 0, the numerator also must equal zero.

But the numerator is the total differential of u, and hence $u=a$ is an integral of the system (2).

Therefore, in order that $u=a$ may be an integral of (1), it is necessary and sufficient that $u=a$ be an integral of the system (2) Art. 99, and conversely.

Moreover, any function whatever of the u and the v of Art. 99 is also a solution of (1); for example,

$$\phi(u, v)=\phi(a, b)=c, \text{ or } \phi(u, v)=0,$$

which is equally general, since c can be involved in the arbitrary function. This can be verified directly. Hence, if

$$u=a, \ v=b,$$

be independent integrals of the system (2) Art. 99,

$$\phi(u, v)=0$$

is the general expression for the integrals of these equations. The arbitrary functional relation may just as well be written in the form $u=f(v)$. This deduction will be used in Art. 115.

101. Geometrical meaning of simultaneous differential equations of the first order and the first degree involving three variables.

Equations (1) or (2) Art. 99 will determine, for each point (x, y, z), definite values of $\frac{dz}{dx}$ and $\frac{dy}{dx}$; that is, these differential equations determine a particular direction at each point in space. Therefore, if a point moves, so that at any moment the co-ordinates of its position and the direction cosines of its line of motion (these cosines being proportional to dx, dy, dz, and hence to P, Q, R, by (2) Art. 99) satisfy the differential equations, then this point must pass through each position in a particular direction. Suppose that a moving point P starts at any point and moves in the direction determined for this point by the differential equations to a second point at an infinitesimal distance; thence, under the same conditions to a third point; thence to a fourth point, and so on; then P will describe a curve in space, whose direction at any one of its points and the co-ordinates of this point will satisfy the given differential equations. If P start from another point, not on the last curve, it will describe another curve; through every point of space there will thus pass a definite curve, whose equation satisfies the given differential equations. These curves are the intersections of the two surfaces which are represented by the two equations forming the solutions; for, these two equations together determine the points and the ratios of dx, dy, dz, thereat which satisfy the differential equations. Moreover, the curves are doubly infinite in number; for they are the intersections of the surfaces represented by the independent integrals $u = a$, $v = b$, and each of these equations contains an arbitrary constant which can take an infinite number of values.

Thus, the locus of the points that satisfy the differential equations of Ex. 1, Art. 99, is the curves, doubly infinite in number, which are the intersections of the system of planes whose equation is

$$y = az,$$

with the system of spheres whose equation is

$$x^2 + y^2 + z^2 = bz;$$

and the locus of the points that satisfy the equations of Ex. 2, Art. 99, is the curves which are the intersections of all the planes represented by

$$lx + my + nz = c,$$

c having an infinite number of values, with all the spheres

$$x^2 + y^2 + z^2 = k^2,$$

k having an infinite number of values.

102. Single differential equations that are integrable. Condition of integrability. The equation

$$Pdx + Qdy + Rdz = 0 \tag{1}$$

has an integral

$$u = a, \tag{2}$$

when there is a function u whose total differential du is equal to the first member of (1), or to that member multiplied by a factor. If (1) have an integral (2), then, since

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz,$$

P, Q, R, must be proportional to $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$; that is,

$$\mu P = \frac{\partial u}{\partial x},$$

$$\mu Q = \frac{\partial u}{\partial y},$$

$$\mu R = \frac{\partial u}{\partial z}.$$

These three conditions can be reduced to one involving the coefficients P, Q, R, and their derivatives. On differentiating the first of these three equations with respect to y and z, the second with respect to z and x, and the third with respect to x and y, there results,

$$\left.\begin{aligned}P\frac{\partial\mu}{\partial y}+\mu\frac{\partial P}{\partial y}=\frac{\partial^2 u}{\partial x\,\partial y}=Q\frac{\partial\mu}{\partial x}+\mu\frac{\partial Q}{\partial x}\\ Q\frac{\partial\mu}{\partial z}+\mu\frac{\partial Q}{\partial z}=\frac{\partial^2 u}{\partial y\,\partial z}=R\frac{\partial\mu}{\partial y}+\mu\frac{\partial R}{\partial y}\\ R\frac{\partial\mu}{\partial x}+\mu\frac{\partial R}{\partial x}=\frac{\partial^2 u}{\partial z\,\partial x}=P\frac{\partial\mu}{\partial z}+\mu\frac{\partial P}{\partial z}\end{aligned}\right\};$$

whence, on rearranging, comes

$$\left.\begin{aligned}\mu\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=Q\frac{\partial\mu}{\partial x}-P\frac{\partial\mu}{\partial y}\\ \mu\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=R\frac{\partial\mu}{\partial y}-Q\frac{\partial\mu}{\partial z}\\ \mu\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=P\frac{\partial\mu}{\partial z}-R\frac{\partial\mu}{\partial x}\end{aligned}\right\}.$$

On multiplying the first of the last three equations by R, the second by P, the third by Q, and adding, there is obtained,

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0, \qquad (3)$$

the relation that must exist between the coefficients of (1) when it has an integral (2).

Conversely, when relation (3) is satisfied, equation (1) has an integral;* and hence (3) is the necessary and sufficient condition that (1) be integrable. It is called the condition or criterion of integrability of the single differential equation (1); and is easily remembered, for P, Q, R, x, y, z, appear in it in a regular cyclical order.

103. Method of finding the solution of the single integrable equation. Suppose that the condition for the integrability of

$$Pdx+Qdy+Rdz=0 \qquad (1)$$

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* For proof of this, see Note H.

is satisfied; a method has now to be devised for finding its solution. Pdx can only come from the terms of the integral that contain x, Qdy from the terms that contain y, and Rdz from the terms that contain z. Hence the integral of (1) is found in the following way:

Consider any one of the variables, say z, as constant, that is, take $dz = 0$, and integrate the equation

$$Pdx + Qdy = 0. \tag{2}$$

Put the arbitrary constant of integration that must appear in the integral of (2) equal to an arbitrary function of z. This is allowable because the arbitrary constant in the integral of (2) is a constant only with respect to x and y. On differentiating the integral just found, with respect to x, y, and z, and comparing the result with (1), it will be possible to determine the constant appearing in the integral of (2) as a particular function of z.

Equations which are homogeneous in x, y, z, like those in Art. 9 in x, y, are always integrable. The initial step in solving these equations is the substitution of zu for x, and of zv for y.*

NOTE. That an equation of the form

$$Pdx + Qdy + Rdz + Tdt + \cdots = 0,$$

involving more than three variables, may have an integral, condition (3) Art. 102 must hold for the coefficients of all the terms taken by threes. All the conditions thus formed are, however, not independent.†

Ex. 1. Solve $(y + z)dx + (z + x)dy + (x + y)dz = 0.$

Here, the condition of integrability is satisfied.

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* See Johnson, *Differential Equations*, Art. 250.

† See Johnson, *Differential Equations*, Arts. 252–254; Forsyth, *Differential Equations*, Arts. 163, 164. For a complete proof of these propositions, see Forsyth, *Theory of Differential Equations*, Part I., pp. 4–12.

Suppose that z is a constant, then $dz = 0$, and the equation becomes

$$(y + z)dx + (z + x)dy = 0;$$

and this on integration yields

$$(y + z)(z + x) = A = \phi(z).$$

Differentiation with respect to x, y, z, gives

$$(y + z)dx + (z + x)dy + (x + y)dz + 2\,z\,dz - \frac{d\phi}{dz}dz = 0.$$

Comparison with the given equation shows that

$$2\,z\,dz - d\phi = 0;$$

whence $$\phi(z) = z^2 + c^2.$$

Therefore, $$(y + z)(z + x) = z^2 + c^2;$$

or, reducing, $xy + yz + zx = c^2$ is a solution of the given equation.

This example can be solved more easily by rearranging the terms in the following way:

$$x\,dy + y\,dx + y\,dz + z\,dy + z\,dx + x\,dz = 0,$$

where the integral is seen at a glance to be

$$xy + yz + zx = c^2.$$

It is well to try to obtain the integral by rearranging the terms, before having recourse to the regular method.

Ex. 2. Solve $\dfrac{x\,dx + y\,dy + z\,dz}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + \dfrac{z\,dx - x\,dz}{x^2 + z^2} + 3\,ax^2dx + 2\,by\,dy + c\,dz = 0.$

Here there is no need to apply the condition of integrability, for the several parts are obviously exact differentials; the integral is immediately seen to be

$$\sqrt{x^2 + y^2 + z^2} + \tan^{-1}\frac{x}{z} + ax^3 + by^2 + cz = k.$$

Ex. 3. Solve $(y + z)dx + dy + dz = 0.$

Ex. 4. Solve $zy\,dx = zx\,dy + y^2dz.$

Ex. 5. Solve $(2\,x^2 + 2\,xy + 2\,xz^2 + 1)dx + dy + 2\,z\,dz = 0.$

Ex. 6. Solve $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0.$

104. Geometrical meaning of the single differential equation which is integrable. Suppose that the equation

$$Pdx + Qdy + Rdz = 0 \tag{1}$$

satisfies the condition of integrability, and that its solution is

$$F(x, y, z) = C. \tag{2}$$

Equation (2) represents a single infinity of surfaces, there being one arbitrary constant. This constant can be determined so that (2) will represent the surface which passes through any given point of space. If a point is moving upon this surface in any direction, the co-ordinates of its position and the direction cosines of its path at any moment, which are proportional to dx, dy, dz, satisfy (1), since (2) is the integral of (1). Also for each point (x, y, z) there will be an infinite number of values of $\frac{dx}{dz}$ and $\frac{dy}{dz}$ which will satisfy (1); therefore, a point that is moving in such a way that its co-ordinates and the direction cosines of its path always satisfy (1) can pass through any point in an infinity of directions. But, when passing through any point, it must remain on the particular surface represented by the integral (2) which passes through the point; hence all the possible curves, infinite in number, which it can describe through that point must lie on that surface.

It has been shown in Art. 101 that a point which is moving subject to the restrictions imposed by the two equations (1) Art. 99 can describe only one curve through any one point; on the other hand, a point that is moving subject to the restriction of a single integrable equation can describe an infinity of curves through that point; all the latter curves, however, lie upon the same surface.

For example, a point passing through the point (1, 2, 3) in such a direction as to satisfy the equations of Ex. 1, Art. 99, must move along the intersection of the plane having the equation

$$3y = 2z$$

and the sphere whose equation is

$$3(x^2+y^2+z^2)=14z.$$

A point moving so as to satisfy the equation of Ex. 1, Art. 103, can pass through (1, 2, 3) in an infinity of directions, but all these possible paths lie upon the surface having the equation

$$xy+yz+zx=11.$$

105. The locus of $Pdx+Qdy+Rdz=0$ is orthogonal to the locus of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$. The equation

$$Pdx+Qdy+Rdz=0 \tag{1}$$

means, geometrically, that a straight line whose direction cosines are proportional to dx, dy, dz, is perpendicular to a line whose direction cosines are proportional to P, Q, R.* Therefore, a point that is moving subject to the condition expressed by (1) must go in a direction *at right angles* to a line whose direction cosines are proportional to P, Q, R.

On the other hand, the equations

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{2}$$

mean, geometrically, that a straight line whose direction cosines are proportional to dx, dy, dz, is parallel to a line whose direction cosines are proportional to P, Q, R. Therefore, a point that is moving subject to the conditions expressed by (2) must go in a direction *parallel* to a line whose direction cosines are proportional to P, Q, R. Therefore, the curves traced out by points that are moving subject to the condition (1) are orthogonal to the curves traced out by points that are moving subject to the conditions (2). The former curves are any of the curves upon the surfaces represented by (1); therefore the curves represented by (2) are normal to the surfaces represented by (1). If (1) be not integrable, there is no family of surfaces which is orthogonal to all the lines that form the locus of equations (2).

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* C. Smith, *Solid Geometry*, Art. 24.

The principle deduced in this article will be employed in Art. 118 of the next chapter.

106. The single differential equation which is non-integrable. When the condition of integrability is not satisfied for

$$Pdx + Qdy + Rdz = 0, \tag{1}$$

there is no single relation between x, y, z, as, for example, $f(x, y, z) = c$, that will satisfy (1).

If, however, there be assumed some integral relation,

$$\phi(x, y, z) = a, \tag{2}$$

which on differentiation gives the differential relation

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0, \tag{3}$$

two integral relations can be found which together satisfy (1) and (3), this being the case discussed in Art. 99. Of course, (2) is one of these relations.

Suppose that $$F(x, y, z) = b \tag{4}$$

is a relation which with (2) forms the complete solution of equations (1) and (3). In Art. 101 it was shown that the locus of the complete solution of (1) and (3) consists of the curves of intersection of (2) and (4); hence, geometrically, this solution of (1) amounts to finding the curves satisfying (1) that lie on the surfaces represented by (2).

Ex. The equation

$$(1) \quad x\,dx + y\,dy + c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dz = 0$$

is one for which the condition of integrability is not satisfied. Suppose that the relation

$$(2) \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

be assumed.

In virtue of (2), (1) may be written in the form

$$(3)\quad x\,dx + y\,dy + z\,dz = 0\,;$$

whence

$$(4)\quad x^2 + y^2 + z^2 = c^2.$$

Thus (2) with (4) gives a solution of (1). Had a relation other than (2) been assumed, a co-relation other than (4) would have been obtained. The geometric interpretation is, that the lines upon the ellipsoid represented by (2) which satisfy (1), have been determined; and have been found to be the intersections of the family of spheres whose equation is (4) with that ellipsoid.

EXAMPLES ON CHAPTER XI.

1. $\dfrac{dx}{dt} + 2\dfrac{dy}{dt} - 2x + 2y = 3e^t$

$3\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2x + y = 4e^{2t}.$

2. $4\dfrac{dx}{dt} + 9\dfrac{dy}{dt} + 2x + 31y = e^t$

$3\dfrac{dx}{dt} + 7\dfrac{dy}{dt} + x + 24y = 3.$

3. $2\dfrac{d^2y}{dx^2} - \dfrac{dz}{dx} - 4y = 2x$

$2\dfrac{dy}{dx} + 4\dfrac{dz}{dx} - 3z = 0.$

4. $\dfrac{dx}{dt} + 4x + 3y = t$

$\dfrac{dy}{dt} + 2x + 5y = e^t.$

5. $t\,dx = (t - 2x)dt$

$t\,dy = (tx + ty + 2x - t)dt.$

6. $x^2dx^2 + y^2dy^2 - z^2dz^2 + 2xy\,dx\,dy = 0.$

7. $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0.$

8. $(y^2 + yz + z^2)dx + (x^2 + xz + z^2)dy + (x^2 + xy + y^2)dz = 0.$

9. $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0.$

10. $z(y + z)dx + z(u - x)dy + y(x - u)dz + y(y + z)du = 0.$

11. $(2x + y^2 + 2xz)dx + 2xy\,dy + x^2dz = du.$

12. $z\,dz + (x - a)dx = \{h^2 - z^2 - (x - a)^2\}^{\frac{1}{2}}\,dy.$

13. $\dfrac{d^2x}{dt^2} + 4x + y = te^{3t}$

$\dfrac{d^2y}{dt^2} + y - 2x = \cos^2 t.$

14. $\dfrac{d^2x}{dt^2} + m^2y = 0$

$\dfrac{d^2y}{dt^2} - m^2x = 0.$

15. $$\frac{dx}{dt}=ny-mz$$
$$\frac{dy}{dt}=lz-nx$$
$$\frac{dz}{dt}=mx-ly.$$

16. $\frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}$, where
$$X=ax+by+cz+d$$
$$Y=a'x+b'y+c'z+d'$$
$$Z=a''x+b''y+c''z+d''.$$

17. Show that the integrals of the system

$$\frac{dx}{dt}=ax+by+c,\ \frac{dy}{dt}=a'x+b'y+c',$$

are
$$(a+m_1a')(x+m_1y)+c+m_1c'=A_1e^{(a+m_1a')t},$$
$$(a+m_2a')(x+m_2y)+c+m_2c'=A_2e^{(a+m_2a')t},$$

where m_1, m_2, are the roots of

$$a'm^2+(a-b')m-b=0;$$

and obtain a similar solution for the system

$$\frac{d^2x}{dt^2}=ax+by,\quad \frac{d^2y}{dt^2}=a'x+b'y.$$

(Johnson, *Diff. Eq.*, Ex. 16, p. 269.)

18. Find the equation of the path described by a particle subject only to the action of gravity, after being projected with an initial velocity v_0 in a direction inclined at an angle ϕ to the horizon.

19. Determine the path of a projectile in a resisting medium such as air when the retardation is c times the velocity, given that the initial velocity is v_0 in a direction inclined at an angle ϕ to the horizon.

20. Find the path described by a particle acted upon by a central force, the force being directly proportional to the distance of the particle.

21. The two fundamental equations of the simple analytical theory of the transformer are

$$R_1i_1+L_1\frac{di_1}{dt}+M\frac{di_2}{dt}=e_1,$$

$$R_2i_2+L_2\frac{di_2}{dt}+M\frac{di_1}{dt}=0,$$

where i_1, i_2, denote the currents, R_1, R_2, the resistances, L_1, L_2, the coefficients of self-induction of the primary and secondary currents respectively, e_1 the impressed primary electromotive force, and M the mutual induction.

Show that, e, i_1, i_2, and t being variable, the differential equations for the primary and secondary currents respectively are,

$$(L_1L_2 - M^2)\frac{d^2i_1}{dt^2} + (L_1R_2 + L_2R_1)\frac{di_1}{dt} + R_1R_2i_1 = R_2e_1 + L_2\frac{de_1}{dt},$$

$$(L_1L_2 - M^2)\frac{d^2i_2}{dt^2} + (L_1R_2 + L_2R_1)\frac{di_2}{dt} + R_1R_2i_2 = -M\frac{de_1}{dt}.$$

(Bedell, *The Principles of the Transformer*, Chap. VI.)

22. The general equations for electromotive forces in the two circuits of a transformer with capacities c_1 and c_2 being

$$e = f(t) = \frac{\int i_1dt}{c_1} + R_1c_1 + L_1\frac{di_1}{dt} + M\frac{di_2}{dt},$$

$$0 = \frac{\int i_2dt}{c_2} + R_2c_2 + L_2\frac{di_2}{dt} + M\frac{di_2}{dt},$$

where e, i_1, i_2, t, are variable, show that the differential equations for the primary and secondary currents are

$$(L_1L_2 - M^2)\frac{d^4i_1}{dt^4} + (R_1L_2 + R_2L_1)\frac{d^3i}{dt^3} + \left(\frac{L_1}{c_2} + \frac{L_2}{c_1} + R_1R_2\right)\frac{d^2i_1}{dt^2}$$
$$+\left(\frac{R_1}{c_2} + \frac{R_2}{c_1}\right)\frac{di_1}{dt} + \frac{1}{c_1c_2}i_1 = \frac{1}{c_2}f'(t) + R_2f''(t) + L_2f'''(t).$$

$$(L_1L_2 - M^2)\frac{d^4i_2}{dt^4} + (R_1L_2 + R_2L_1)\frac{d^3i_2}{dt^3} + \left(\frac{L_1}{c_2} + \frac{L_2}{c_1} + R_1R_2\right)\frac{d^2i_2}{dt^2}$$
$$+\left(\frac{R_1}{c_2} + \frac{R_2}{c_1}\right)\frac{di_2}{dt} + \frac{1}{c_1c_2}i_2 = -Mf'''(t).$$

(Bedell, *The Principles of the Transformer*, Chap. XI.)

CHAPTER XII.

PARTIAL DIFFERENTIAL EQUATIONS.

107. Definitions. Partial differential equations are those which contain one or more partial derivatives, and must, therefore, be concerned with at least two independent variables.*

The derivation of partial differential equations will be discussed in Arts. 108, 109; equations of the first order will be considered in Arts. 110–123; and those of the second and higher orders in the remaining part of the chapter. These equations, excepting the ones treated in Arts. 117, 134–136, will involve only three variables. In what follows, x and y will usually be taken as the independent variables, and z as dependent; the partial differential coefficients $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$, will be denoted by p and q respectively.

108. Derivation of a partial differential equation by the elimination of constants. Partial differential equations can be derived in two ways: (a) by the elimination of arbitrary constants from a relation between x, y, z, and (b) by the elimination of arbitrary functions of these variables. To illustrate (a) take

$$\phi(x, y, z, a, b) = 0 \qquad (1)$$

a relation between x, y, z, the latter variable being dependent upon x and y. In order to eliminate the two constants a, b, two more equations are required. These equations can be obtained from (1) by differentiation with respect to x and y; they will be

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* Equations with partial derivatives were at first studied by D'Alembert (see p. 173), and Euler (see p. 64), in connection with problems of physics.

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial z}p=0,\quad \frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial z}q=0.$$

By means of these three equations, a and b can be eliminated, and there will appear a relation of the form

$$F(x, y, z, p, q)=0, \tag{2}$$

a partial differential equation of the first order.

In (1) the number of constants eliminated is just equal to the number of independent variables, and an equation of the first order arises. If the number of constants to be eliminated is greater than the number of independent variables, equations of the second and higher orders will, in general, be derived. The following examples will illustrate this. In these examples, z is to be taken as the dependent variable.

Ex. 1. Form a partial differential equation by the elimination of the constants h and k from

$$(x-h)^2+(y-k)^2+z^2=c^2.$$

Differentiating with respect to x and y,

$$x-h+zp=0,$$
$$y-k+zq=0.$$

Substituting the values of $x-h$, $y-k$ from the last two equations in the given equation,

$$z^2(p^2+q^2+1)=c^2.$$

Ex. 2. Form the partial differential equation corresponding to

$$z=ax+by+ab.$$

Ex. 3. Eliminate a and b from $z=a(x+y)+b$.

Ex. 4. Eliminate a and b from $z=ax+a^2y^2+b$.

Ex. 5. Eliminate a and b from $z=(x+a)(y+b)$.

Ex. 6. Form a partial differential equation by eliminating a, b, c from

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

109. Derivation of a partial differential equation by the elimination of an arbitrary function. To illustrate (*b*) of Art. 108, suppose that u and v are functions of x, y, z, and that there is a relation between u and v of the form

$$\phi(u, v) = 0, \tag{1}$$

where ϕ is arbitrary. The relation may also be expressed in the form $u = f(v)$, where f is arbitrary. It is now to be shown, that, on the elimination of the arbitrary function ϕ from (1), a partial differential equation will be formed; and, moreover, that this equation will be *linear*, that is, it will be of the first degree in p and q.

Differentiation of (1) with respect to each of the independent variables x and y gives

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0,$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0.$$

Elimination of $\frac{\partial \phi}{\partial u}$, $\frac{\partial \phi}{\partial v}$, from these two equations results in

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right);$$

and this can be rearranged in the form

$$Pp + Qq = R, \tag{2}$$

where

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y},$$

$$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z},$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

Thus, from (1), which involves an arbitrary function ϕ, a partial differential equation (2) has been obtained, which does not contain ϕ and is linear in p and q.

When the given relation between x, y, z contains two arbitrary functions, the partial differential equation derived therefrom will, except in particular cases, involve partial derivatives of an order higher than the second.*

Ex. 1. Eliminate the arbitrary function from $z = e^{ny}\phi(x-y)$.

Differentiating with respect to x, $p = e^{ny}\phi'(x-y)$.

Differentiating with respect to y, $q = ne^{ny}\phi(x-y) - e^{ny}\phi'(x-y)$;

and, therefore, $q = nz - p$,

that is, $p + q = nz$.

Ex. 2. Form a partial differential equation by eliminating the arbitrary function from $z = F(x^2+y^2)$.

Ex. 3. Eliminate the function ϕ from $lx + my + nz = \phi(x^2+y^2+z^2)$.

Ex. 4. Eliminate the function f from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Ex. 5. Eliminate the arbitrary functions f and ϕ from

$$z = f(x+ay) + \phi(x-ay).$$

PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

110. The integrals of the non-linear equation: the complete and particular integrals. In Art. 108 it was shown how the partial differential equation

$$F(x, y, z, p, q) = 0 \tag{1}$$

may be derived from

$$\phi(x, y, z, a, b) = 0. \tag{2}$$

Suppose, now, that (2) has been derived from (1), by one of the methods hereafter shown; then the solution (2), which has as many arbitrary constants as there are independent variables, is called the complete integral of (1).

A particular integral of (1) is obtained by giving particular values to a and b in (2).

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* See Edwards, *Differential Calculus*, Arts. 509–514; Williamson, *Differential Calculus*, Arts. 315–319; Johnson, *Differential Equations*, Arts. 299–301.

111. The singular integral. The locus of all the points whose co-ordinates with the corresponding values of p and q satisfy (1) Art. 110, is the doubly infinite system of surfaces represented by (2). The system is doubly infinite, because there are two constants, a and b, each of which can take an infinite number of values. Since the envelope of all the surfaces represented by $\phi(x, y, z, a, b) = 0$ is touched at each of its points by some one of these surfaces, the co-ordinates of any point on the envelope with the p and the q belonging to the envelope at that point must satisfy (1); and, therefore, the equation of the envelope is an integral of (1). The equation of the envelope of the surfaces represented by (2) is obtained in the following way:*

Eliminate a and b between the three equations,

$$\phi(x, y, z, a, b) = 0,$$

$$\frac{d\phi}{da} = 0,$$

$$\frac{d\phi}{db} = 0;$$

and the relation thus found between x, y, z is the equation of the envelope. This relation is called the singular integral; it differs from a particular integral in that it is not contained in the complete integral; that is, it is not obtained from the complete integral by giving particular values to the constants. (Compare Arts. 32, 33.)

112. The general integral. Suppose that in (2) Art. 110, one of the constants is a function of the other, say $b = f(a)$, then this equation becomes

$$\phi(x, y, z, a, f(a)) = 0, \tag{1}$$

which represents one of the families of surfaces included in the system represented by (2). The equation of the envelope of the family of surfaces represented by (1) will also satisfy (1) Art. 110, for reasons similar to those given in the case of the singular integral.

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* For proof see C. Smith, *Solid Geometry*, Arts. 211-215; W. S. Aldis, *Solid Geometry*, Chap. X.

Moreover, this equation will be different from that of the envelope of all the surfaces, and it is not a particular integral. It is called the general integral; and it is found by eliminating a between

$$\phi(x, y, z, a, f(a)) = 0,$$

and

$$\frac{d\phi}{da} = 0.$$

These two equations together represent a curve, namely, the curve of intersection of two consecutive surfaces of the system $\phi(x, y, z, a, f(a)) = 0$. The envelope of the family of surfaces, being the locus of the ultimate intersections of the surfaces belonging to the family, that is, of the intersections of consecutive surfaces, contains this curve to which the name *characteristic* of the envelope has been given. Hence the general integral may be defined as the locus of the characteristics.

Other relations may appear in the process of deriving the singular and the general integrals from the complete integral, but it is beyond the scope of this work to discuss such relations. When one has performed the operations necessary to find the singular and the general integrals, he should test his result by trying whether it satisfies the differential equation. (Compare Arts. 33–38.)

In the case of every equation, the general integral and the singular integral, as well as the complete integral, must be indicated or the equation is not considered to be fully solved. The complete integral is to be found first, and from it the other two are to be derived.* It is evident that the locus of the singular integral will be the envelope of the loci of all the other integrals, of the general as well as of the complete.

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* The distinction between the three kinds of integrals of partial differential equations was made by Lagrange in *Memoirs of the Berlin Academy*, 1772, 1774.

Ex. In Ex. 1, Art. 108, the differential equation

$$z^2(p^2+q^2+1)=c^2 \tag{1}$$

was derived from

$$(x-h)^2+(y-k)^2+z^2=c^2. \tag{2}$$

The latter equation, which contains two arbitrary constants, is the complete integral of the former; it represents the doubly infinite system of spheres of radius c, whose centres are in the xy plane.

A particular integral of (1) is obtained by giving h and k particular values in (2); thus,

$$(x-2)^2+(y-3)^2+z^2=c^2$$

is a particular integral.

The singular integral of (1) is the equation that represents the envelope of these spheres; it is obtained by eliminating h and k from (2) by means of the relations derived by differentiating (2) with respect to h and k.

The differentiation gives

$$x-h=0,$$

$$y-k=0;$$

on substituting these values in (2), h and k are eliminated, and there results the equation

$$z=\pm c.$$

This satisfies equation (1), and, therefore, is the singular integral. It represents the two planes that are touched by all the spheres represented by (2).

Suppose, now, that one of the constants is made a function of the other, say, that

$$k=h.$$

Then the centres, since their co-ordinates have that relation, are restricted to the straight line $y=x$ in the xy plane; and of the system of spheres representing (2) there will be chosen a particular family, namely,

$$(x-h)^2+(y-h)^2+z^2=c^2. \tag{3}$$

The envelope of this family is the tubular surface, in this case a cylinder, which is generated by a sphere of radius c, when its centre moves along the line $y=x$. The equation of this envelope is a general integral; it is found by eliminating h from (3) by means of the relation obtained by differentiating (3) with respect to h.

The differentiation gives $x-h+y-h=0$,

whence $$h=\tfrac{1}{2}(x+y).$$

Substituting this value of h in (3),

$$x^2 + y^2 - 2xy + 2z^2 = 2c^2,$$

which is a general integral.

If the relation between the constants were assumed to be

$$k^2 = 4ah,$$

the corresponding general integral would be the equation of the tubular surface generated by a sphere of radius c, whose centre moves along the parabola $y^2 = 4ax$ in the xy plane.

113. The integral of the linear equation.* In Art. 109 it was shown that from an arbitrary functional relation

$$\phi(u, v) = 0 \tag{1}$$

there is derived, by the elimination of the function ϕ, a linear partial differential equation

$$Pp + Qq = R. \tag{2}$$

Suppose that (1) has been derived from (2); then $\phi(u, v) = 0$ is called the general solution of (2). Since ϕ is an arbitrary function, the solution (1) is more general than another solution of (2) that merely contains arbitrary constants. For instance, Ex. 2, Art. 109, shows that the general solution of

$$yp - xq = 0$$

is

$$z = F(x^2 + y^2),$$

where F denotes an arbitrary function. The arbitrary function F may take various forms, as,

$$z = a(x^2 + y^2)^2 + b(x^2 + y^2),$$

$$z = a \sin(x^2 + y^2) + b,$$

etc.,

which are all solutions of the differential equation, and are included in the general solution above.

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* The student will find it of great advantage to read C. Smith, *Solid Geometry*, Arts. 216–226; W. S. Aldis, *Solid Geometry*, Arts. 142–151, in connection with this and following articles.

114. Equation equivalent to the linear equation. The type of a partial differential equation which is linear in p and q is

$$Pp + Qq = R, \tag{1}$$

P, Q, R being functions of x, y, z.

Suppose that $$u = a$$

is any relation that satisfies (1); differentiation with respect to x and y gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p = 0,$$

$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q = 0;$$

whence $$p = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial z}}, \; q = -\frac{\dfrac{\partial u}{\partial y}}{\dfrac{\partial u}{\partial z}}.$$

Substitution of these values of p and q in (1) changes it to

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0. \tag{2}$$

Therefore, if $u = a$ be an integral of (1), $u = a$ also satisfies (2). Conversely, if $u = a$ be an integral of (2), it is also an integral of (1). This can be seen by dividing by $\frac{\partial u}{\partial z}$ and substituting p and q for the values above. *Therefore equation* (2) *can be taken as equivalent to equation* (1).

115. Lagrange's solution of the linear equation. In Art. 100 it was shown that

$$\phi(u, v) = 0$$

is a general integral of (2) Art. 114 when $u = a$, $v = b$ are independent integrals of the system of equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Hence the following rule may be given:

To obtain an integral of the linear equation of the form

$$Pp + Qq = R,$$

find two independent integrals of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}; \tag{3}$$

let them be $u = a$ and $v = b$;

then $$\phi(u, v) = 0,$$

where ϕ is an arbitrary function, is an integral of the partial differential equation.

Instead of $\phi(u, v) = 0$, there can with equal generality be written $u = f(v)$, where f denotes an arbitrary function.

This is known as Lagrange's solution of the linear equation;[*] the auxiliary equations (3) are called Lagrange's equations; and the curves of intersection of the surfaces represented by the integrals of (3) are called Lagrangean lines.

116. Verification of Lagrange's solution. The truth of Lagrange's solution may also be shown in the following way. Form the differential equations corresponding to $u = a$ and $v = b$, by eliminating the arbitrary constants a and b; this gives

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0,$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0,$$

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[*] Joseph Louis Lagrange (1736–1813) was one of the greatest mathematicians that the world has ever seen. He wrote much on differential equations, and the theory of the linear partial equation was first given by him. He discussed the case of three variables and gave the solution in a memoir in the Berlin Academy of Sciences in 1772; he treated singular solutions in a memoir of 1774; and in memoirs of 1779 and 1785 he gave a generalised method applicable to equations having any number of variables. See footnote, page 40.

whence $$\frac{dx}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z}-\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}}=\frac{dy}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}}=\frac{dz}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y}-\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}.$$

But, in Art. 109, it was found that the equation derived from $\phi(u, v) = 0$ by eliminating ϕ is

$$\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

Comparison shows that these equations have the forms

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

and $$Pp+Qq=R, \text{ respectively.}$$

Ex. 1. Solve $xzp + yzq = xy$.

Dividing by xyz, $$\frac{p}{y}+\frac{q}{x}=\frac{1}{z};$$

forming the auxiliary equations,

$$y\,dx = x\,dy = z\,dz.$$

Integrating the equation formed by the first two terms,

$$\frac{y}{x}=c.$$

Also $y\,dx + x\,dy = 2z\,dz$; whence $z^2 - xy = c$.

Therefore, the solution is $z^2 - xy = \phi\left(\dfrac{y}{x}\right)$, or $f\left(z^2 - xy, \dfrac{y}{x}\right) = 0$.

Ex. 2. Solve $p + q = \dfrac{z}{a}$.

Ex. 3. Solve $(mz - ny)p + (nx - lz)q = ly - mx$.

Ex. 4. Solve $x^2p + y^2q = z^2$.

Ex. 5. Solve $\dfrac{y^2zp}{x} + xzq = y^2$.

117. The linear equation involving more than two independent variables. If there be n functions $u_1, u_2, \cdots, u_n$, of $n + 1$ variables $z, x_1, x_2, \cdots, x_n$, z being dependent and the other variables independent, then the arbitrary function ϕ can be eliminated from

$$\phi(u_1, u_2, \cdots, u_n) = 0 \tag{1}$$

by an extension of the method used in Art. 109. The result will be a linear partial differential equation of the form

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \cdots + P_n\frac{\partial z}{\partial x_n} = R. \tag{2}$$

Moreover, on forming the differential equations corresponding to $u_1 = c_1$, $u_2 = c_2$, $\cdots$, $u_n = c_n$, by eliminating the constants $c_1, c_2, \cdots, c_n$, and proceeding as in Art. 116, there will be obtained

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}. \tag{3}$$

Hence the following rule may be given:

In order to deduce the general integral of the partial differential equation (2), write down the auxiliary equations (3), and find n independent integrals of this system of equations; let these integrals be

$$u_1 = c_1,\ u_2 = c_2, \cdots, u_n = c_n;$$

then
$$\phi(u_1, u_2, \cdots, u_n) = 0,$$

where ϕ denotes an arbitrary function is the integral of the given equation.

Suppose that $u = c$ is an integral of (2); then

since
$$\frac{\partial z}{\partial x_i} = -\frac{\dfrac{\partial u}{\partial x_i}}{\dfrac{\partial u}{\partial z}},\ (i = 1, 2, \cdots, n),$$

equation (2) can take the equivalent form

$$P_1\frac{\partial u}{\partial x_1} + P_2\frac{\partial u}{\partial x_2} + \cdots + P_n\frac{\partial u}{\partial x_n} + R\frac{\partial u}{\partial z} = 0. \tag{4}$$

Ex. 1. Solve $(t+y+z)\frac{\partial t}{\partial x}+(t+x+z)\frac{\partial t}{\partial y}+(t+x+y)\frac{\partial t}{\partial z}=x+y+z.$

The auxiliary equations are

$$\frac{dt}{x+y+z}=\frac{dx}{y+z+t}=\frac{dy}{z+t+x}=\frac{dz}{t+x+y};$$

whence, $$\frac{dt+dx+dy+dz}{3(t+x+y+z)}=\frac{dt-dx}{x-t}=\cdots;$$

from this, $$\log(t+x+y+z)^{\frac{1}{3}}=\log\frac{c_1}{x-t};$$

hence, $$(x-t)(t+x+y+z)^{\frac{1}{3}}=c_1;$$

similarly, $$(y-t)(t+x+y+z)^{\frac{1}{3}}=c_2,$$

and $$(z-t)(t+x+y+z)^{\frac{1}{3}}=c_3.$$

Hence the solution is

$$\phi\{(x-t)u,\ (y-t)u,\ (z-t)u\}=0,$$

where $$u=(t+x+y+z)^{\frac{1}{3}}.$$

Ex. 2. Solve $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=xyz.$

118. Geometrical meaning of the linear partial differential equation. In Art. 105 it was shown that the curves whose equations are integrals of

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \tag{1}$$

are at right angles to the system of surfaces whose equation satisfies

$$Pdx+Qdy+Rdz=0. \tag{2}$$

Suppose that $$u=a,\ v=b$$

are any pair of independent integrals of (1). Let a take a particular value, say a_1. The surface represented by $u=a_1$ is intersected by the system of surfaces whose equation is $v=b$, in an infinite number of curves, a curve for each one of the infinite number of values that b can have. Thus $u=a_1$ represents a locus which passes through, or upon which lie, curves infinite in number, that are orthogonal to the surfaces represented by (2).[†] Therefore, since the general integral of

$$Pp + Qq = R \tag{3}$$

is an arbitrary function of integrals of equations (1), any integral of (3) passes through a system of lines that are orthogonal to the surfaces forming the locus of (2); and hence the surfaces represented by (3) are orthogonal to the surfaces represented by (2).

[*] **119. Special methods of solution applicable to certain standard forms.**

There are a few standard forms to which many equations are reducible, and which can be integrated by methods that are sometimes shorter than the general method which will be shown in Art. 123. These forms will now be discussed.

Standard I. To this standard belong equations that involve p and q only; they have the form

$$F(p, q) = 0. \tag{1}$$

A solution of this is evidently

$$z = ax + by + c,$$

if a and b be such that $F(a, b) = 0$; that is, solving the last equation for b, if $b = f(a)$. The complete integral then is

$$z = ax + yf(a) + c. \tag{2}$$

The general integral is obtained by putting $c = \phi(a)$, where ϕ denotes an arbitrary function, and eliminating a between

$$z = ax + yf(a) + \phi(a),$$

and

$$0 = x + yf'(a) + \phi'(a).$$

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[*] Arts. 119–122 closely follow Forsyth, *Differential Equations*, Arts. 191–196.

[†] When such surfaces exist. See Arts. 104–106.

The singular integral is obtained by eliminating a and c between the complete integral (2) and the equations formed by differentiating (2) with respect to a and c; that is, between

$$z = ax + yf(a) + c,$$
$$0 = x + yf'(a),$$
$$0 = 1;$$

the last equation shows that there is no singular integral.

Ex. **1.** Solve (1) $p^2 + q^2 = m^2$.

The solution is $z = ax + by + c$, if $a^2 + b^2 = m^2$.

Therefore, the complete solution is

$$(2) \quad z = ax + \sqrt{m^2 - a^2}\, y + c.$$

To find the general integral, put $c = f(a)$;

then $$z = ax + \sqrt{m^2 - a^2}\, y + f(a);$$

differentiate with respect to a,

$$0 = x - \frac{a}{\sqrt{m^2 - a^2}} y + f'(a);$$

and eliminate a by means of these two equations.

A developable surface is the envelope of a plane whose equation contains only one variable parameter.* Therefore, the general integral in this case represents a developable surface. In particular, if c or $f(a)$ be chosen equal to zero, then the result obtained by eliminating a is

$$(3) \quad z^2 = m^2(x^2 + y^2).$$

The complete integral (2) represents a doubly infinite system of planes; the particular integral obtained by putting c equal to zero represents a singly infinite system of planes passing through the origin; and the general integral (3) represents the cone which is the envelope of the latter system of planes.

Ex. **2.** Solve (1) $$x^2p^2 + y^2q^2 = z^2.$$

This may be written $\left(\frac{x\partial z}{z\partial x}\right)^2 + \left(\frac{y\partial z}{z\partial y}\right)^2 = 1$. Put $\frac{dx}{x} = dX$, $\frac{dy}{y} = dY$, $\frac{dz}{z} = dZ$; whence $X = \log x$, $Y = \log y$, $Z = \log z$; the equation then

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* See C. Smith, *Solid Geometry*, Art. 221.

becomes
$$\left(\frac{\partial Z}{\partial X}\right)^2+\left(\frac{\partial Z}{\partial Y}\right)^2=1,$$
which comes under Standard I.

From the preceding example the complete integral is
$$Z=aX+\sqrt{1-a^2}\,Y+\log c;$$
hence, $z=cx^a y^{\sqrt{1-a^2}}$, which is the complete integral of (1). The singular integral is $z=0$; the general integral is to be found in the usual way.

Ex. **3.** Solve $3p^2-2q^2=4pq$.

Ex. **4.** Solve $q=e^{-\frac{p}{a}}$.

Ex. **5.** Solve $pq=k$.

120. Standard II. To this standard belong equations analogous to Clairaut's; they have the form
$$z=px+qy+f(p,q). \tag{1}$$
That the solution is
$$z=ax+by+f(a,b) \tag{2}$$
can easily be verified. This is the complete integral, since it contains two arbitrary constants. It represents a doubly infinite system of planes.

In order to obtain the general integral, put $b=\phi(a)$, where ϕ denotes an arbitrary function; then
$$z=ax+y\phi(a)+f\{a,\phi(a)\};$$
differentiate this with respect to a,
$$0=x+y\phi'(a)+f'(a),$$
and eliminate a between these equations.

In order to obtain the singular integral, differentiate
$$z=ax+by+f(a,b)$$
with respect to a and b, thereby getting the equations
$$0=x+\frac{df}{da},\quad 0=y+\frac{df}{db};$$
and eliminate a and b between these three equations.

Ex. 1. Solve $z = px + qy + pq$.

The complete integral is

$$z = ax + by + ab.$$

In order to find the singular integral, differentiate with respect to a and b; this gives

$$0 = x + b,$$
$$0 = y + a;$$

elimination of a and b by means of these equations gives $z = -xy$.

The general integral is the a eliminant of

$$z = ax + yf(a) + af(a),$$
$$0 = x + yf'(a) + af'(a) + f(a),$$

where f denotes an arbitrary function.

Ex. 2. Solve $z = px + qy - 2\sqrt{pq}$.

121. Standard III. To this standard belong equations that do not contain x or y; they have the form

$$F(z, p, q) = 0. \tag{1}$$

Put X for $x + ay$, where a is an arbitrary constant, and assume

$$z = f(x + ay) = f(X)$$

for a trial solution; then

$$p = \frac{dz}{dX} \cdot \frac{\partial X}{\partial x} = \frac{dz}{dX}, \quad q = \frac{dz}{dX} \cdot \frac{\partial X}{\partial y} = a\frac{dz}{dX}.$$

Substitution in (1) gives

$$F\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) = 0, \tag{2}$$

which is an ordinary differential equation of the first order.

The solution of (2) gives an expression of the form

$$\frac{dz}{dX} = \phi(z, a),$$

whence,
$$\frac{dz}{\phi(z, a)} = dX;$$

integrating,
$$f(z, a) = X + b,$$

and hence, $$x + ay + b = f(z, a)$$
is the complete integral.

The general and the singular integrals are to be found as before.

This method of solving equations of Standard III. can be formulated in the following rule:

Substitute ap for q, and change p to $\frac{dz}{dX}$, X being equal to $x + ay$; then solve the resulting ordinary differential equation between z and X.

Ex. 1. Solve (1) $z^2(p^2 + q^2 + 1) = c^2$.

On putting ap for q, changing p to $\frac{dz}{dX}$, and separating the variables, (1) becomes

$$\sqrt{a^2+1}\frac{z\,dz}{\sqrt{c^2-z^2}} = dX.$$

Integrating, $$-\sqrt{a^2+1}\sqrt{c^2-z^2} = X + b;$$

squaring, and substituting for X its value $x + ay$,

$$(2) \quad (a^2+1)(c^2-z^2) = (x + ay + b)^2.$$

This is the complete integral of (1), since it contains two independent arbitrary constants a and b.

Differentiate (2) with respect to a and b, and eliminate a and b; there results

$$z^2 = c^2,$$

which satisfies (1), and is thus the singular solution.

In order to find a general integral, substitute for b some function of a, and eliminate a from the equation.

In particular, on putting
$$b = -ak - h,$$
(2) becomes
$$(3) \quad (a^2+1)(c^2-z^2) = \{x - h + a(y-k)\}^2. \qquad (3)$$

Differentiation with respect to a gives the equation

$$2a(c^2-z^2) = 2(y-k)\{x - h + a(y-k)\},$$

which in virtue of (3) can be put in the form

$$(4) \quad \{x - h + a(y-k)\}\{a(x-h) - (y-k)\} = 0. \qquad (4)$$

On eliminating a from (3) by means of the first component equation of (4), there appears the equation

$$z^2 = c^2;$$

and on eliminating a by means of the second component equation, there comes

$$(x-h)^2 + (y-k)^2 + z^2 = c^2. \tag{5}$$

The general integral is thus made up of the last two equations, which represent two parallel planes and a sphere. The planes and sphere form the envelope of the cylinders represented by (3). Equation (5) may also be regarded as a complete integral, if h and k be taken as arbitrary constants. (See Ex. 1, Art. 108 and Art. 112.)

Ex. **2**. Solve $q^2y^2 = z(z - px)$.

This may be written

$$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right);$$

and putting dY for $\frac{dy}{y}$, dX for $\frac{dx}{x}$, (whence $Y = \log y$ and $X = \log x$), the latter equation becomes

$$\left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right),$$

which belongs to Standard III.

Ex. **3**. Solve $9(p^2z + q^2) = 4$.

Ex. **4**. Solve $p(1 + q^2) = q(z - a)$.

Ex. **5**. Solve $pz = 1 + q^2$.

122. Standard IV. To this standard belong equations that have the form

$$f_1(x, p) = f_2(y, q). \tag{1}$$

In some partial differential equations in which the variable z does not appear, it happens that the terms containing p and x can be separated from those containing q and y; the equation then has the form (1).

Put each of these equal expressions equal to an arbitrary constant a, thus,

$$f_1(x, p) = a, \quad f_2(y, q) = a;$$

and solve these equations for p and q, thus obtaining

$$p = F_1(x, a), \ q = F_2(y, a).$$

Integration of the last two equations gives

$$z = \int F_1(x, a)dx + \text{a quantity independent of } x,$$

and $$z = \int F_2(y, a)dy + \text{a quantity independent of } y.$$

These are included in, or are equivalent to

$$z = \int F_1(x, a)\,dx + \int F_2(y, a)\,dy + b,$$

where b is an arbitrary constant.

This is the complete integral, since it contains two arbitrary constants; the general integral and the singular integral, if existing, are to be found as before.

Ex. 1. Solve $q - p + x - y = 0$.

Separating q and y from p and x,

$$q - y = p - x.$$

Write $$q - y = p - x = a;$$

hence $p = x + a$ and $q = y + a$; and therefore the complete integral is

$$2z = (x + a)^2 + (y + a)^2 + b.$$

There is no singular integral; the general integral is given by the elimination of a between

$$2z = (x + a)^2 + (y + a)^2 + f(a)$$

and $$0 = 2(x + a) + 2(y + a) + f'(a),$$

f being an arbitrary function.

Ex. 2. Solve $p^2 - q^2 = \dfrac{x - y}{z}$.

Hence $zp^2 - x = zq^2 - y$. Put dZ for $z^{\frac{1}{2}}dz$.

Ex. 3. Solve $q = 2yp^2$.

Ex. 4. Solve $\sqrt{p} + \sqrt{q} = 2x$.

Ex. 5. Solve $p^2 + q^2 = x + y$.

Ex. 6. Solve $z^2(p^2 + q^2) = x^2 + y^2$.

123. General method of solution.* It will be remembered that, in order to solve some of the ordinary differential equations of the first order in Arts. 24–29, another differential relation was deduced; and by means of the two differential relations, that were thus at command, the derivative was eliminated and a solution obtained. The general method of solving partial equations of the first order will be found to present some points of analogy to the method employed in the articles referred to.

Take the partial differential equation

$$F(x, y, z, p, q) = 0. \tag{1}$$

Since z depends upon x and y, it follows that

$$dz = p\,dx + q\,dy. \tag{2}$$

Now if another relation can be found between x, y, z, p, q, such as

$$f(x, y, z, p, q) = 0, \tag{3}$$

then p and q can be eliminated; for the values of p and q deduced from (1) and (3) can be substituted in (2). The integral of the ordinary differential equation thus formed involving x, y, z, will satisfy the given equation (1); for the values of p and q that will be derived from it are the same as the values of p and q in (1).

A method of finding the needed relation (3) must now be devised. Assume (3) for the unknown relation between x, y, z, p, q, which, in connection with (1), will determine values of p and q that will render (2) integrable. On differentiating (1) and (3) with respect to x and y, the following equations appear:

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* This method, commonly known as Charpit's method, in which the non-linear partial equation is connected with a system of linear ordinary equations, is due partly to Lagrange, but was perfected by Charpit. It was first fully set forth in a memoir presented by Charpit to the Paris Academy of Sciences, June 30, 1784. The author died young, and the memoir was never published.

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0,$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0,$$

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0,$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0.$$

The elimination of $\frac{\partial p}{\partial x}$ between the first pair of these equations gives

$$\left(\frac{\partial F}{\partial x}\frac{\partial f}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial x}\right)+p\left(\frac{\partial F}{\partial z}\frac{\partial f}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial z}\right)+\frac{\partial q}{\partial x}\left(\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}-\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)=0;$$

and the elimination of $\frac{\partial q}{\partial y}$ between the second pair gives

$$\left(\frac{\partial F}{\partial y}\frac{\partial f}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial y}\right)+q\left(\frac{\partial F}{\partial z}\frac{\partial f}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial z}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial F}{\partial p}\frac{\partial f}{\partial q}-\frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)=0.$$

On adding the first members of these two equations, the last bracketed terms cancel each other, since

$$\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial p}{\partial y};$$

hence, adding and re-arranging,

$$\left(\frac{\partial F}{\partial x}+p\frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial p}+\left(\frac{\partial F}{\partial y}+q\frac{\partial F}{\partial z}\right)\frac{\partial f}{\partial q}+\left(-p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial z}$$
$$+\left(-\frac{\partial F}{\partial p}\right)\frac{\partial f}{\partial x}+\left(-\frac{\partial F}{\partial q}\right)\frac{\partial f}{\partial y}=0. \qquad (4)$$

This is a linear equation of the first order, which the auxiliary function f of equation (3) must satisfy. This form has been considered in Art. 117, and its integrals are the integrals of

$$\frac{dp}{\dfrac{\partial F}{\partial x}+p\dfrac{\partial F}{\partial z}}=\frac{dq}{\dfrac{\partial F}{\partial y}+q\dfrac{\partial F}{\partial z}}=\frac{dz}{-p\dfrac{\partial F}{\partial p}-q\dfrac{\partial F}{\partial q}}=\frac{dx}{\dfrac{-\partial F}{\partial p}}$$
$$=\frac{dy}{\dfrac{-\partial F}{\partial q}}=\frac{df}{0}. \tag{5}$$

Any of the integrals of (5) satisfy (4); if such an integral involve p or q, it can be taken for the required second relation (3). Of course, the simpler the integral involving p or q, or both p and q that is derived from (5), the easier will be the subsequent labor in finding the solution of (1).

This method is applicable to all partial differential equations of the first order; but it is often better to enquire whether the equation to be solved is reducible to one of the standard forms discussed in Arts. 119–122. The reduction and the subsequent integration by one of the special methods is generally, but not always, less laborious than the integration by the general method. By applying the general method to the linear equation and the standard forms, the integrals obtained in the preceding sections are easily obtained.*

Ex. 1. Solve

(1) $p(q^2+1)+(b-z)q=0.$

Here equations (5) Art. 123 reduce to

(2) $\dfrac{dp}{pq}=\dfrac{dq}{q^2}=\dfrac{dz}{3pq^2+p+(b-z)q}=\dfrac{dx}{q^2+1}=\dfrac{dy}{-z+b+2pq}.$

The third fraction, by virtue of the given equation, reduces to $\dfrac{dz}{2pq^2}$.

From the first two fractions, there comes, on integration,

$$q=ap,$$

where a is an arbitrary constant.

This and the original equation determine the values of p and q; namely,

$$p=\frac{\sqrt{a(z-b)-1}}{a},\quad q=\sqrt{a(z-b)-1}.$$

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* See Forsyth, *Differential Equations*, Arts. 203–207; Johnson, *Differential Equations*, Arts. 288–293.

Substitution of these values in

$$dz = p\,dx + q\,dy$$

gives
$$dz = \left(\frac{dx}{a} + dy\right)\sqrt{a(z-b)-1},$$

where the variables are separable; this on integration gives

$$2\sqrt{a(z-b)-1} = x + ay + b.$$

There is no singular solution; the general solution is obtained in the usual way.

This equation comes under Standard III., and the ratios chosen from (2) give the relation $q = ap$, which is used in the special method. Had there been chosen the equation formed by another pair of ratios from (2), say from

$$\frac{dq}{q^2} = \frac{dx}{q^2+1},$$

another complete integral would have been obtained; namely,

$$(z-b)\left\{\frac{x+\alpha}{2} - \sqrt{\left(\frac{x+\alpha}{2}\right)^2 + 1}\right\} + y + \beta = 0.$$

Ex. **2**. Solve $z = pq$ by the general method.

Ex. **3**. Solve $(p^2 + q^2)y = qz$.

Ex. **4**. Solve the linear equation and the standard forms by the general method.

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND AND HIGHER ORDERS.

124. Partial equations of the second order. In this and the following articles,* a few of the simplest forms of partial differential equations of the second order will be briefly considered; hardly more will be done, however, than to indicate the methods of obtaining their solutions. Some of these equations are of the highest importance in physical investigations.

In what follows, z being the dependent variable, and x and y the independent, r, s, t will denote the second derivatives:

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* In connection with these articles read the introductory chapter of W. E. Byerly, *Fourier's Series and Spherical Harmonics*.

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x\, \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}.$$

There will be discussed linear equations only; that is, equations of the first degree in r, s, t, which are thus of the form

$$Rr + Ss + Tt = V,$$

where R, S, T, V are functions of x, y, z, p, q. The complete solutions of these equations will contain two arbitrary functions.* In Art. 125 will be given some examples of equations that are readily integrable, the special method of solution necessary being easily seen; and in Art. 126 will be given a general method of solution.

125. Examples readily solvable. It is to be remembered that x and y, being independent, are constant with regard to each other in integration and differentiation.

Ex. 1. Solve $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{x}{y} + a.$

Writing it $$\frac{dp}{dy} = \frac{x}{y} + a,$$

integration with regard to y gives

$$p = x \log y + ay + \phi_1(x),$$

the constant with regard to y being possibly a function of x.

Integrating the last equation with regard to x gives

$$z = \int \{x \log y + ay + \phi_1(x)\} dx,$$

$$= \frac{x^2}{2} \log y + axy + \phi(x) + \psi(y),$$

the constant with regard to x being possibly a function of y.

Ex. 2. Solve $\dfrac{\partial^2 z}{\partial x\, \partial y} + \dfrac{\partial z}{\partial x} f(x) = F(y).$

Rewrite it, $$\frac{dp}{dy} + pf(x) = F(y).$$

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* See Art. 109, Ex. 5.

This equation is linear in p, and x is constant with regard to y; hence integration gives

$$p = e^{-yf(x)}\left[\int e^{yf(x)}F(y)dy + \phi(x)\right];$$

and integration of this with regard to x gives

$$z = \int\left\{e^{-yf(x)}\left[\int e^{yf(x)}F(y)dy + \phi(x)\right]\right\}dx + \psi(y).$$

Ex. **3.** $ar = xy.$

Ex. **4.** $xr = (n-1)p.$

126. General method of solving $Rr + Ss + Tt = V$. On writing the total differentials of p and q,

$$dp = r\,dx + s\,dy,$$

$$dq = s\,dx + t\,dy,$$

the elimination of r and t by means of these from the given equation,

$$Rr + Ss + Tt = V, \tag{1}$$

gives $(R\,dp\,dy + T\,dq\,dx - V\,dx\,dy) - s(R\,dy^2 - S\,dx\,dy + T\,dx^2) = 0.$

If any relation between x, y, z, p, q will make each of the bracketed expressions vanish, this relation will satisfy (1).

From
$$\left.\begin{array}{r} R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0 \\ R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \end{array}\right\}^{*} \tag{2}$$

and
$$dz = p\,dx + q\,dy,$$

it may be possible to derive either one or two relations between x, y, z, p, q called intermediary integrals, and therefrom to deduce the general solution of (1). For an investigation of the conditions under which this equation admits an intermediary integral, and for the deduction of the way of finding the

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[*] These are called Monge's equations, after Gaspard Monge (1746–1818), the inventor of descriptive geometry, who tried to integrate equations of the form $Rr + Ss + Tt = 0$, in 1784, and succeeded in some simple cases. The method of this article is also called by his name.

general integral see Forsyth, *Differential Equations*, Arts. 228–239. The statement of the method of solution derived from this investigation is contained in the following rule:

Form first the equation

$$R\,dy^2 - S\,dx\,dy + T\,dx^2 = 0, \tag{3}$$

and resolve it, supposing the first member not a complete square, into the two equations

$$dy - m_1 dx = 0, \quad dy - m_2 dx = 0. \tag{4}$$

From the first of these, and from the equation

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy, = 0 \tag{5}$$

combined if necessary with $dz = p\,dx + q\,dy$, obtain two integrals $u_1 = a$, $v_1 = b$; then

$$u_1 = f_1(v_1),$$

where f_1 is an arbitrary function, is an intermediary integral.

From the second of the equations (4), in the same way, obtain another pair of integrals, $u_2 = a$, $v_2 = b$; then

$$u_2 = f_2(v_2)$$

is another intermediary integral, f_2 being arbitrary

To deduce the final integral, either of these intermediary integrals may be integrated; and this must be done when $m_1 = m_2$. When m_1 and m_2 are unequal, the two intermediate integrals are solved for p and q, and their values substituted in

$$dz = p\,dx + q\,dy,$$

which, when integrated, gives the complete integral.

Ex. 1. Solve $r - a^2t = 0$. (This equation is solved by another method in Art. 128.)

Here the subsidiary equations (4) and (5) are

$$(1) \quad dy + a\,dx = 0, \; dy - a\,dx = 0,$$

$$(2) \quad dp\,dy - a^2 dx\,dq = 0.$$

Hence $$y + ax = c_1, \; y - ax = c_2.$$

Combining the first of equations (1) with (2),

$$dp + a\,dq = 0, \text{ whence } p + aq = c'_1 = F_1(y + ax);$$

combining the second of (1) with (2),

$$dp - a\,dq = 0, \text{ whence } p - aq = c'_2 = F_2(y - ax).$$

From the last two integrals

$$p = \tfrac{1}{2}[F_1(y + ax) + F_2(y - ax)],$$

and

$$q = \frac{1}{2\,a}[F_1(y + ax) - F_2(y - ax)].$$

Substitution of these values of p and q in $dz = p\,dx + q\,dy$, gives, on rearranging terms,

$$dz = \frac{1}{2\,a}[F_1(y + ax)(dy + a\,dx) - F_2(y - ax)(dy - a\,dx)],$$

which is exact. Integration gives

$$z = \phi(y + ax) + \psi(y - ax).$$

The equation in this example, $\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial y^2} = 0$, is a very important one in mathematical physics. It is called the equation of vibrating cords, sometimes D'Alembert's equation, from the name of the geometer who first integrated it in 1747.* It appears in considering the vibrations of a stretched elastic string, t being the time, y being measured along the string, and u being the small transversal displacement of any point. This equation also gives the law of small oscillations in a thin tube of air, for instance, in an organ-pipe. The functions ϕ and ψ that appear in the general solution are to be determined from the given initial conditions.

Ex. 2. $ps - qr = 0$. Ex. 3. $x^2r + 2\,xys + y^2t = 0$.

127. The general linear partial equation of an order higher than the first. A partial differential equation, which is linear with respect to the dependent variable and its derivatives, is of the form

$$A_0\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial x^{n-1}\,\partial y} + \cdots + A_n\frac{\partial^n z}{\partial y^n} + B_0\frac{\partial^{n-1} z}{\partial x^{n-1}} + \cdots$$
$$+ M\frac{\partial z}{\partial x} + N\frac{\partial z}{\partial y} + Pz = f(x, y), \quad (1)$$

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* Jean-le-Rond D'Alembert (1717–1783), who first announced in 1743 the principle in dynamics that bears his name, was one of the pioneers in the study of differential equations.

where the coefficients are constants or functions of x and y. On putting D for $\frac{\partial}{\partial x}$ and D' for $\frac{\partial}{\partial y}$, this may be written

$$(A_0D^n + A_1D^{n-1}D' + \cdots + A_nD'^n + \cdots + MD + ND' + P)z$$
$$= f(x, y), \tag{2}$$

or briefly, $$F(D, D')z = f(x, y). \tag{3}$$

As in the case of linear equations between two variables (see Art. 49), the complete solution consists of two parts, the *complementary function* and the *particular integral*, the complementary function being the solution of

$$F(D, D')z = 0. \tag{4}$$

Also, if $z = z_1,\ z = z_2,\ \cdots,\ z = z_n$ be solutions of (4),

$$z = c_1z_1 + c_2z_2 + \cdots + c_nz_n$$

is also a solution.

Other analogies between linear partial and linear ordinary equations, especially in methods of solving, will be observed in the following articles.

128. The homogeneous equation with constant coefficients: the complementary function. All the derivatives appearing in this equation are of the same order, and it is of the form

$$(A_0D^n + A_1D^{n-1}D' + \cdots + A_nD'^n)z = f(x, y). \tag{1}$$

If it be assumed that $z = \phi(y + mx)$, differentiation will show that

$$Dz = m\phi'(y + mx),\ D^nz = m^n\phi^{(n)}(y + mx),\ D'^nz = \phi^{(n)}(y + mx),$$

and, in general, that

$$D^rD'^sz = m^r\phi^{(r+s)}(y + mx).$$

Therefore, the substitution of $\phi(y + mx)$ for z in the first member of (1) gives $(A_0m^n + A_1m^{n-1} + \cdots + A_n)\phi^{(n)}(y + mx)$. This is zero, and consequently, $\phi(y + mx)$ is a part of the complementary function if m is a root of

$$A_0m^n + A_1m^{n-1} + \cdots + A_n = 0, \tag{2}$$

which may be called *the auxiliary equation.*

Suppose that the n roots of (2) are $m_1, m_2, \cdots, m_n$, then the complementary function of (1) is

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \cdots + \phi_n(y + m_nx),$$

where the functions ϕ are arbitrary. The factors of the coefficient of z in (1) corresponding to these roots are $D - m_1D'$, $D - m_2D'$, $\cdots$, $D - m_nD'$; and these are easily shown to be commutative. (Compare Arts. 50, 54.)

Since $e^{mxD'}\phi(y) = (1 + mxD' + \frac{m^2x^2}{2\,!}D'^2 + \cdots)\,\phi(y)$,

$$= \phi(y) + mx\,\phi'(y) + \frac{m^2x^2}{2\,!}\phi''(y) + \cdots,$$

$$= \phi(y + mx),$$

the part of the C.F. corresponding to a root m of (2) may be written $e^{mxD'}\phi(y)$.

Ex. 1. $\frac{\partial^2z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0.$ (See Ex. 1, Art. 126.)

Here (2) is $m^2 - a^2 = 0$, whence m has the values $+a$, $-a$. Hence the solution is $z = \phi(y + ax) + \psi(y - ax)$.

Ex. 2. Find the C.F. of $\frac{\partial^2z}{\partial x^2} + 3\frac{\partial^2z}{\partial x\,\partial y} + 2\frac{\partial^2z}{\partial y^2} = x + y.$

Ex. 3. Find the C.F. of $\frac{\partial^2z}{\partial x^2} - \frac{\partial^2z}{\partial x\,\partial y} - 6\frac{\partial^2z}{\partial y^2} = xy.$

129. Solution when the auxiliary equation has repeated or imaginary roots. As in the case of equations between two variables (see Arts. 51, 52), further investigation is required when the roots of (2) Art. 128 are multiple or imaginary.

The equation corresponding to two repeated roots m is

$$(D - mD')(D - mD')\,z = 0.$$

On putting v for $(D - mD')\,z$, this becomes $(D - mD')\,v = 0$, of which the solution is $v = \phi(y + mx)$. Hence

$$(D - mD')z = \phi(y + mx).$$

The Lagrangean equations of this linear equation of the first order are

$$dx = -\frac{dy}{m} = \frac{dz}{\phi(y + mx)}.$$

The integrals of these equations are $y + mx = a$, $z = x\phi(a) + b$; and hence,

$$z = x\phi(y + mx) + \psi(y + mx).$$

By proceeding in this way it can be shown that when a root m is repeated r times, the corresponding part of the complementary function is

$$x^{r-1}\phi_1(y+mx) + x^{r-2}\phi_2(y+mx) + \cdots + x\phi_{r-1}(y+mx) + \phi_r(y+mx).$$

When the roots of (2) Art. 128 are imaginary, the corresponding part of the solution can be made to take a real form.*

Ex. $\dfrac{\partial^3 z}{\partial x^3} - 3\dfrac{\partial^3 z}{\partial x^2\,\partial y} + 3\dfrac{\partial^3 z}{\partial x\,\partial y^2} - \dfrac{\partial^3 z}{\partial y^3} = 0.$

130. The particular integral. Equation (1) Art. 128 being expressed by $F(D, D')z = \phi(x, y)$, the particular integral will be denoted by $\dfrac{1}{F(D, D')}\phi(x, y)$, $\dfrac{1}{F(D, D')}V$ being defined as that function which gives V when it is operated upon by $F(D, D')$. (Compare Art. 57.)

By Art. 128,

$$\frac{1}{F(D, D')}\phi(x, y) = \frac{1}{D - m_1D'}\cdot\frac{1}{D - m_2D'}\cdots\frac{1}{D - m_nD'}\phi(x, y). \quad (1)$$

It is easily shown that it follows from the definition of $\dfrac{1}{F(D, D')}V$ that these factors are commutative. The value of $\dfrac{1}{D - mD'}\phi(x, y)$ will now be indicated. For this purpose, it is first necessary to evaluate $(D - mD')\,\phi(x, y)$. From the latter part of Art. 128 it follows that

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* See Johnson, *Differential Equations*, Art. 319; Merriman and Woodward, *Higher Mathematics*, Chap. VII., Art. 25.

$$e^{-mxD'}\phi(x, y) = \phi(x, y - mx);$$

therefore, $$De^{-mxD'}\phi(x, y) = D\phi(x, y - mx).$$

Direct differentiation shows that

$$De^{-mxD'}\phi(x, y) = e^{-mxD'}(D - mD')\phi(x, y).$$

From equating the second members of the last two equations, and operating upon these members with $e^{mxD'}$, it follows that

$$(D - mD')\phi(x, y) = e^{mxD'}D\phi(x, y - mx).$$

That a similar formula

$$\frac{1}{D - mD'}\phi(x, y) = e^{mxD'}\frac{1}{D}\phi(x, y - mx) \tag{2}$$

holds true for the inverse operator is easily verified. For, the application of $D - mD'$ to both sides of (2) gives

$$\phi(x, y) = (D - mD')e^{mxD'}\frac{1}{D}\phi(x, y - mx),$$

$$= (D - mD')e^{mxD'}\psi(x, y),$$

on putting $\psi(x, y)$ for $\frac{1}{D}\phi(x, y - mx)$; and, therefore, by Art. 128,

$$\phi(x, y) = (D - mD')\psi(x, y + mx).$$

But the second member of the last equation is also the result that would be obtained by putting $y + mx$ for y in $D\psi(x, y)$ after the differentiation had been performed; and this would be $\phi(x, y)$ from the definition of ψ given above. Hence $\frac{1}{D - mD'}\phi(x, y)$ can be evaluated by the following rule, which is the verbal expression of (2): form the function $\phi(x, y - mx)$, integrate this with respect to x, and in the integral obtained, change y into $y + mx$.

The value of the second member of (1) is obtained by applying the operations indicated by the factors, in succession, beginning at the right. Methods shorter than this general method can be employed in certain cases, which are referred to and exemplified in Art. 132. Ex. 2 also shows such a case.

Ex. 1. Find the particular integral of Ex. 2, Art. 128.

The particular integral

$$=\frac{1}{D^2+3\,DD'+2\,D'^2}(x+y)=\frac{1}{D+2\,D'}\cdot\frac{1}{D+D'}(x+y)$$

$$=\frac{1}{D+2\,D'}e^{-xD'}\frac{1}{D}(2\,x+y)=\frac{1}{D+2\,D'}(x^2+x\cdot\overline{y-x})=\frac{1}{D+2\,D'}xy$$

$$=e^{-2xD'}\frac{1}{D}x(y+2\,x)=\frac{2\,x^3}{3}+\frac{x^2}{2}(y-2\,x)=\frac{x^2y}{2}-\frac{x^3}{3}.$$

Ex. 2. Evaluate $\frac{1}{F(D,\,D')}\phi(ax+by)$. In this case a short method can be used in finding the integral.

Since $F(D,\,D')=D^nF\left(\frac{D'}{D}\right)$, and $\frac{D'}{D}\phi(ax+by)=\frac{b}{a}$, and consequently $F\left(\frac{D'}{D}\right)\phi(ax+by)=F\left(\frac{b}{a}\right)$, it follows that

$$\frac{1}{F(D,\,D')}\phi(ax+by)=\frac{1}{D^nF\left(\frac{D'}{D}\right)}\cdot\phi(ax+by)=\frac{1}{D^n}\frac{1}{F\left(\frac{b}{a}\right)}\phi(ax+by)$$

$$=\frac{1}{F\left(\frac{b}{a}\right)}\int\int\cdots\int\phi(ax+by)\,(dx)^n.$$

When $\frac{b}{a}$ is a root of $F\left(\frac{D'}{D}\right)=0$, then $F\left(\frac{D'}{D}\right)=\left(\frac{D'}{D}-\frac{b}{a}\right)\psi\left(\frac{D'}{D}\right)$, and the integral is $\frac{1}{\psi\left(\frac{b}{a}\right)}\cdot\frac{1}{D\left(\frac{D'}{D}-\frac{b}{a}\right)}\int\int\cdots\int\phi(ax+b)\,(dx)^{n-1}$; the latter expression can be evaluated by the general rule.

Ex. 3. Find the particular integral of $\frac{\partial^2u}{\partial x^2}-a^2\frac{\partial^2u}{\partial y^2}=x^2$.

Ex. 4. Find the particular integral of Ex. 3, Art. 128.

131. The non-homogeneous equation with constant coefficients the complementary function. In order to find the complementary function of (3) Art. 127, that is, the solution of

$$F(D, D')z = 0, \tag{1}$$

first assume $z = ce^{hx+ky}$. (This procedure is like that of Art. 50.) The substitution of this value of z in $F(D, D')z$ gives $cF(h, k)e^{hx+ky}$. This is zero if

$$F(h, k) = 0; \tag{2}$$

and then $z = ce^{hx+ky}$ is a part of the complementary function. The solution of (2) for k will give values $f_1(h), f_2(h), \cdots, f_r(h)$, if D' is of degree r in (1). The part of the solution of (1) corresponding to $k = f_1(h)$ is $\Sigma c_1 e^{hx+f_1(h)y}$, Σ indicating the infinite series obtained by giving c and h all possible arbitrary values; hence the general solution corresponding to all the values of k is

$$z = \Sigma c_1 e^{hx+f_1(h)y} + \Sigma c_2 e^{hx+f_2(h)y} + \cdots + \Sigma c_r e^{hx+f_r(h)y}.$$

This solution can be put in a simpler form when $f(h)$ is linear in h, that is, when $k = ah + b$. In particular this is true of the homogeneous equation, which is, of course, a special case of (1). Exs. 2, 3 illustrate these remarks. Equally well may (2) be solved for h in terms of k, and another form of the solution will be obtained, as in Exs. 1, 2.

Ex. 1. $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{\partial^2 z}{\partial y^2} = 0.$

Here (2) is $h^3 - k^2 = 0$, whence $k = h^{\frac{3}{2}}$, and thus the solution is $z = \Sigma ce^{hx+h^{\frac{3}{2}}y}$, where c and h are arbitrary. Particular integrals are obtained by giving h particular values; for example, the values 1, 5, $\frac{3}{4}$ for h give the particular solutions $z = e^{x+y}$, $z = e^{5x+5^{\frac{3}{2}}y}$, $z = e^{\frac{3}{4}x+\frac{3}{8}^{\frac{3}{2}}y}$.

If equation (2) be solved for h, the particular integral is $\Sigma ce^{ky+k^{\frac{2}{3}}x}$.

Ex. 2. $2\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} - \dfrac{\partial^2 z}{\partial y^2} + 6\dfrac{\partial z}{\partial x} + 3\dfrac{\partial z}{\partial y} = 0.$

Here (2) is $2h^2 - hk - k^2 + 6h + 3k = 0$, where the values of k are $-2h$, $h + 3$; hence

$$z = \Sigma c_1 e^{h(x-2y)} + \Sigma c_2 e^{h(x+y)+3y} = \Sigma c_1 e^{h(x-2y)} + e^{3y}\Sigma c_2 e^{h(x+y)}.$$

Since each of these series consists of terms having arbitrary coefficients and exponents, it can be represented by an arbitrary function. Consequently the solution can be represented by

$$z = \phi(x - 2y) + e^{3y}\psi(x + y).$$

The equation above might have been solved for h, the values being $-\frac{k}{2}$, $k - 3$. Hence,

$$z = \Sigma c_1 e^{k\left(y - \frac{x}{2}\right)} + \Sigma e^{k(x+y)-3x} = \phi\left(y - \frac{x}{2}\right) + e^{-3x}\psi(x + y)$$

is another way in which the solution may be written.

Ex. **3**. Solve Ex. 1, Art. 128 by this method. Here the values of h are ak, $-ak$, and hence

$$z = \Sigma c_1 e^{k(y+ax)} + \Sigma c_2 e^{k(y-ax)} = \phi(y + ax) + \psi(y - ax).$$

Ex. **4**. Find the complementary function of

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} - 3\frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} = xy + e^{x+2y}.$$

Ex. **5**. Find the complementary function of

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial z}{\partial y} - z = \cos(x + 2y) + e^{y}.$$

Ex. **6**. Find the complementary function of

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} + 3\frac{\partial z}{\partial y} - 2z = e^{x-y} - x^2 y.$$

132. The particular integral. The particular integral can be obtained in certain cases by methods analogous to those shown in Arts. 60–64. It is easily shown, by the method adopted in Arts. 60–62, that $\frac{1}{F(D, D')} \cdot e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}$; that $\frac{1}{F(D^2, DD', D'^2)} \sin(ax+by) = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by)$, and similarly for the cosine; and that $\frac{1}{F(D, D')} x^r y^s$ can be evaluated by operating upon $x^r y^s$ with $[F(D, D')]^{-1}$ expanded in ascending powers of D and D'.*

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* For a full discussion, see Johnson, *Differential Equations*, Arts. 328–334.

Ex. 1. $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}-2\dfrac{\partial^2 z}{\partial y^2}+2\dfrac{\partial z}{\partial x}+2\dfrac{\partial z}{\partial y}=e^{2x+3y}+\sin(2x+y)+xy.$

The complementary function, found by Art. 131, is

$$\phi(y-x)+e^{-2x}\psi(2x+y).$$

The particular integral is

$$\frac{1}{D^2-DD'-2D'^2+2D+2D'}\{e^{2x+3y}+\sin(2x+y)+xy\}.$$

$$\frac{1}{D^2-DD'-2D'^2+2D+2D'}e^{2x+3y}=\frac{e^{2x+3y}}{-10};$$

$$\frac{1}{D^2-DD'-2D'^2+2D+2D'}\sin(2x+y)$$

$$=\frac{1}{2(D+D')}\sin(2x+y)=\frac{1}{2}\frac{D-D'}{D^2-D'^2}\sin(2x+y)$$

$$=-\frac{\cos(2x+y)}{6}.$$

$$\frac{1}{D^2-DD'-2D'^2+2D+2D'}xy=\frac{1}{D+D'}\cdot\frac{1}{D-2D'+2}\cdot xy$$

$$=\frac{1}{2}\cdot\frac{1}{D+D'}\left(1-\frac{D-2D'}{2}+\frac{(D-2D')^2}{4}\right)xy$$

$$=\frac{1}{2}\cdot\frac{1}{D+D'}(xy-\tfrac{1}{2}y+x-1)$$

$$=\frac{1}{2}\cdot\frac{1}{D}\left(1-\frac{D'}{D}\right)(xy-\tfrac{1}{2}y+x-1)=\frac{1}{2D}(xy-\tfrac{1}{2}y-\frac{x^2}{2}+\tfrac{3}{2}x-1)$$

$$=\frac{x}{24}(6xy-6y-2x^2+9x-12).$$

Therefore, the general solution is

$$z=\phi(y-x)+e^{-2x}\psi(2x+y)-\frac{1}{10}e^{2x+3y}-\frac{1}{6}\cos(2x+y)$$

$$+\frac{x}{24}(6xy-6y-2x^2+9x-12).$$

Ex. 2. Solve Exs. 1, 3, 4, Art. 130, by the shorter methods.

Ex. 3. Find the particular integrals of Exs. 4, 5, 6, Art. 131.

133. Transformation of equations. The linear partial differential equation with variable coefficients, like the linear equation between two variables, may sometimes be transformable into one having constant coefficients. In particular, an equation in which the coefficient of any derivative is of a degree in the independent variables equal to the number indicating the order of the derivative, is thus reducible. This is illustrated by Ex. 1. (Compare Arts. 65, 71.)

Ex. **1.** $x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} - y \frac{\partial z}{\partial y} + x \frac{\partial z}{\partial x} = 0.$

On assuming $u = \log x$, $v = \log y$, the equation takes the form

$$\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} = 0,$$

of which the solution is $z = \phi(u + v) + \psi(u - v)$.

The substitution of the values of u, v, gives

$$z = \phi(\log(xy)) + \psi\left(\log \frac{x}{y}\right) = f(xy) + F\left(\frac{x}{y}\right).$$

Ex. **2.** $x^2 \frac{\partial^2 z}{\partial x^2} - yx\, 4 \frac{\partial^2 z}{\partial x\, \partial y} + 4 y^2 \frac{\partial^2 z}{\partial y^2} + 6 y \frac{\partial z}{\partial y} = x^3 y^4.$

Ex. **3.** $\frac{1}{x^2} \frac{\partial^2 z}{\partial x^2} - \frac{1}{x^3} \frac{\partial z}{\partial x} = \frac{1}{y^2} \frac{\partial^2 z}{\partial y^2} - \frac{1}{y^3} \frac{\partial z}{\partial y}.$

134.* **Laplace's equation:** $\nabla^2 v = 0$. The equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0, \qquad (1)$$

usually written $\nabla^2 v = 0,$

and commonly known as Laplace's † equation, is one of the equations most frequently met in investigations in applied mathematics, appearing, as it does, in discussions on mechanics, sound, electricity, heat, etc., especially where the theory of potential is involved.

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* Arts. 134, 135, 136, are merely notes.

† Because it was first given, in 1782, by Pierre Siméon Laplace (1749-1827), one of the greatest of French mathematicians.

For instance, if V be the Newtonian potential due to an attracting mass, at any point $P(x, y, z)$ not forming a part of the mass itself, V satisfies (1);[*] again, if V be the electric potential at any point (x, y, z) where the electrical density is zero, V satisfies (1);[†] and, to give one more instance, if a body be in a state of equilibrium as to temperature, v being the temperature at any point, $\frac{dv}{dt} = 0$, and v satisfies (1). If $f(x, y, z)$ denote any value of v that satisfies (1), $f(x, y, z) = c$ in the first two instances is called an equipotential surface, and in the third an isothermal surface.

On changing to spherical co-ordinates by the transformation

$$x = r \sin\theta \cos\phi,$$
$$y = r \sin\theta \overset{\sin}{\cancel{\cos}} \phi,$$
$$z = r \cos\theta,$$

(1) becomes[‡]

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} + \frac{2}{r}\frac{\partial v}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v}{\partial \phi^2} = 0, \qquad (2)$$

which may be written

$$\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left(r^2 \frac{\partial v}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial \phi^2}\right\} = 0; \qquad (3)$$

and if $\mu = \cos\theta$, it will take the form

$$r\frac{\partial^2 (vr)}{\partial r^2} + \frac{\partial}{\partial \mu}\left\{(1-\mu^2)\frac{\partial v}{\partial \mu}\right\} + \frac{1}{1-\mu^2}\frac{\partial^2 v}{\partial \phi^2} = 0. \qquad (4)$$

The subject of *Spherical Harmonics* is in part concerned with the development of functions that will satisfy this equation.* A homogeneous rational integral algebraic function of x, y, z of the nth degree, that is, a function of the form $r^n f(\theta, \phi)$ in spherical co-ordinates, which is a value of v satisfying (1), is called a *solid spherical harmonic* of the nth degree; and $f(\theta, \phi)$ is called a *surface spherical harmonic* of the nth degree. Spherical harmonics are also known as Laplace's coefficients.†

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[*] B. O. Peirce, *Newtonian Potential Function*, Art. 28; Thomson and Tait, *Natural Philosophy*, Art. 491.

[†] W. T. A. Emtage, *Mathematical Theory of Electricity and Magnetism*, p. 14.

[‡] Todhunter, *Differential Calculus*, Art. 207; Williamson, *Differential Calculus*, Art. 323; Edwards, *Differential Calculus*, Art. 532. The equation as given by Laplace was in the form (2).

If v be independent of ϕ, (3) reduces to

$$r\frac{\partial^2(rv)}{\partial r^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=0. \qquad (5)$$

On putting $v = r^n P$, where P is a function of θ only, and changing the independent variable θ by means of the relation $\mu = \cos\theta$, (5) becomes

$$\frac{d}{d\mu}(1-\mu^2)\frac{dP}{d\mu}+n(n+1)P=0, \qquad (6)$$

which is Legendre's equation, Art. 83. A function that satisfies (6) or (5) is called a *surface zonal harmonic*. A particular class of zonal harmonics is also known as Legendrean coefficients.‡ For a treatment of spherical harmonics, see Byerly, *Fourier's Series and Spherical Harmonics*, Chap. VI., pp. 195–218; and of zonal harmonics, see the same work, Chap. V., pp. 144–194.

In special cases (1) and its solution assume simple forms; two of these will now be shown.

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* See Williamson, *Differential Calculus*, Chap. XXIII., Arts. 332–337; Edwards, *Differential Calculus*, Art. 189; Lamb, *Hydrodynamics*, Ed. 1895, Arts. 82–85; Byerly, *Fourier's Series and Spherical Harmonics*.

† So called after Laplace, who employed them in determining V in a paper bearing the date 1782.

‡ After Legendre, who first introduced them in a paper published in 1785. Legendre's work in this subject, however, was done before that of Laplace (Byerly, *Fourier's Series and Spherical Harmonics*, Chap. IX., p. 267). See Ex. 5, Art. 82.

135. Special cases. In the first instance given in Art. 134, suppose that the attracting mass is a sphere composed of concentric shells, each of uniform density. Here v obviously depends only upon the distance of the point P from the centre of the sphere, and hence (2) Art. 134 reduces to

$$\frac{d^2v}{dr^2}+\frac{2}{r}\frac{dv}{dr}=0, \tag{1}$$

which on integration gives

$$v=A+\frac{B}{r}. \tag{2}$$

Equation (1), in which v depends upon r alone, can be obtained directly from (1) Art. 134 by means of the relation $r^2=x^2+y^2+z^2$. For,

$$\frac{\partial v}{\partial x}=\frac{dv}{dr}\frac{\partial r}{\partial x}=\frac{x}{r}\frac{dv}{dr},$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{1}{r}\frac{dv}{dr}-\frac{x^2}{r^3}\frac{dv}{dr}+\frac{x^2}{r^2}\frac{d^2v}{dr^2};$$

and on finding similar values for $\frac{\partial^2 v}{\partial y^2}$, $\frac{\partial^2 v}{\partial z^2}$, and adding, there results (1).

For the discussion and integration of (1) from the point of view of mechanics, see Thomson and Tait, *Natural Philosophy*, Vol. I., Part II., p. 35.

If the point P in the second instance of Art. 134 be outside of a uniformly electrified sphere and at a distance r from the centre, obviously $\frac{\partial v}{\partial \phi}=0$ and $\frac{\partial v}{\partial \theta}=0$; and equations (1) and (2) follow as before. For the interpretation and application of this result, from the point of view of electricity, see Emtage, *Mathematical Theory of Electricity and Magnetism*, pp. 14, 35, 37.

Again, suppose that the attracting body in the first instance in Art. 134 is made up of infinitely long co-axial cylindrical shells, each of uniform density, the z-axis being the common axis of the cylinder; or, that in the second instance P is a point outside an infinitely long conducting cylinder uniformly charged with electricity, the z-axis being the axis of the cylinder. Since in these cases v depends only upon the distance from the axis of the cylinder, that is, upon $x^2 + y^2$, (1) Art. 134 reduces to

$$r\frac{d^2v}{dr^2} + \frac{dv}{dr} = 0,$$

which on integration gives

$$v = A\log\frac{B}{r}; \text{ or } v = C - A\log r.$$

For discussion of these and other special cases, see the works referred to in the former part of this article, and also B. O. Peirce, *Newtonian Potential Function.*

136. Poisson's equation: $\nabla^2 v = -4\pi\rho$. If in (1) Art. 134 the second member be $-4\pi\rho$, ρ being a function of x, y, z, then there appears the equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -4\pi\rho, \tag{1}$$

which is known as Poisson's equation.* An example of its occurrence is the following: † If ρ be the density of matter at the point (x, y, z) in the first instance in Art. 134, equation (1) Art. 134 takes the above form. In the case of the sphere described in Art. 135, the equation becomes

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} = -4\pi\rho,$$

and the first integral is

$$r^2\frac{dv}{dr} = -4\pi\int_0^r \rho r^2 dr = -M,$$

where M denotes the whole amount of matter within the spherical surface of radius r. In the case of the co-axial cylinders, the equation becomes

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* So called from Siméon Denis Poisson (1781–1840), who thus extended Laplace's equation.

† See Thomson and Tait, *Natural Philosophy*, Art. 491.

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} = -4\pi\rho,$$

and the first integral is

$$r\frac{dv}{dr} = c - 4\pi\int_0^r \rho r dr.$$

EXAMPLES ON CHAPTER XII.

1. $xp + yq = nz.$

2. $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0.$

3. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = az + \frac{xy}{t}.$

4. $(a - x)p + (b - y)q = c - z.$

5. $(y + z)p + (z + x)q = x + y.$

6. $a(p + q) = z.$

7. $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^3 - y^3).$

8. $z - xp - yq = a\sqrt{x^2 + y^2 + z^2}.$

9. $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$

10. $\frac{(y - z)p}{yz} + \frac{(z - x)q}{zx} = \frac{x - y}{xy}.$

11. $\cos(x + y)p + \sin(x + y)q = z.$

12. $p^2 = z^2(1 - pq).$

13. $q = (z + px)^2.$

14. $x^2p^2 = yq^2.$

15. $p^2 + q^2 = npq.$

16. $z - px - qy = c\sqrt{1 + p^2 + q^2}.$

17. $\sqrt{p} + \sqrt{q} = 1.$

18. $q = xp + p^2.$

19. $p(1 + q) = qz.$

20. Find three complete integrals of $pq = px + qy.$

21. $(x^2 + y^2)(p^2 + q^2) = 1.$

22. $pq = x^m y^n z^l.$

23. $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1.$

24. $(y - x)(qy - px) = (p - q)^2.$

25. $(p + q)(px + qy) = 1.$

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26. $xr + p = 9x^2y^2.$

27. $s + \frac{2p}{y} = \frac{1}{y^2}.$

28. $q^2r - 2pqs + p^2t = 0.$

29. $q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0.$

30. $yr = (n - 1)yp + a.$

31. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}f(x) = F(y).$

32. $xr - p = xy.$

33. $p + r = xy.$

34. $s = xy.$

35. $r + (a + b)s + abt = xy.$

36. $(b + cq)^2r - 2(b + cq)(a + cp)s + (a + cp)^2t = 0.$

37. $s + \frac{y}{1 - y^2}p = ay^3.$

38. $r + \frac{y}{x}s = 15xy^2.$

MISCELLANEOUS NOTES.

NOTE A.

A system of ordinary differential equations, of which a part or all is of an order higher than the first, can be reduced to a system of equations of the first order.

Take the single differential equation of order n

$$f\left(\frac{d^ny}{dx^n}, \frac{d^{n-1}y}{dx^{n-1}}, \cdots, \frac{dy}{dx}, y, x\right)=0, \tag{1}$$

and put $\frac{dy}{dx}=y_1$, $\frac{d^2y}{dx^2}=y_2$, $\cdots$, $\frac{d^{n-1}y}{dx^{n-1}}=y_{n-1}$. Then (1) can be replaced by the following system of n equations of the first order,

$$\left.\begin{aligned}
&\frac{dy}{dx}=y_1,\\
&\frac{dy_1}{dx}=y_2,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\frac{dy_{n-2}}{dx}=y_{n-1},\\
&f\left(\frac{dy_{n-1}}{dx}, y_{n-1}, y_{n-2}, \cdots, y_1, y, x\right)=0.
\end{aligned}\right\}$$

Again, suppose that there are two simultaneous equations,

$$f_1\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, z, \frac{dz}{dx}, \frac{d^2z}{dx^2}\right)=0,$$

$$f_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, z, \frac{dz}{dx}, \frac{d^2z}{dx^2}\right)=0.$$

On putting $\frac{dy}{dx}=y_1$, $\frac{d^2y}{dx^2}=y_2$, $\frac{dz}{dx}=z_1$, these two equations can be replaced by the following equivalent system of equations of the first order:

$$\left.\begin{aligned}
&\frac{dy}{dx} = y_1,\\
&\frac{dy_1}{dx} = y_2,\\
&\frac{dz}{dx} = z_1,\\
&f_1\left(x,\ y,\ y_1,\ y_2,\ \frac{dy_2}{dx},\ z,\ z_1,\ \frac{dz_1}{dx}\right) = 0,\\
&f_2\left(x,\ y,\ y_1,\ y_2,\ \frac{dy_2}{dx},\ z,\ z_1,\ \frac{dz_1}{dx}\right) = 0.
\end{aligned}\right\}$$

It is evident that any system of ordinary differential equations can be reduced in this manner to another equivalent system, where there will appear only derivatives of the first order.

NOTE B.

[This Note is supplementary to Art. 1.]

The Existence Theorem.

Following is a proof of the existence of an integral of an equation of the first order.*

Suppose that the differential equation $\phi(y',\ y,\ x) = 0$, where y' stands for $\frac{dy}{dx}$, is put in the form

$$y' = f(x,\ y), \tag{1}$$

which is always possible. This proof is limited to the case where $f(x,\ y)$ is a function which can be represented by a power-series †

$$a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + \cdots + a_ix^my^n + \cdots,$$

in which the a's are all known, since $f(x,\ y)$ is known, and which converges for $|x| \overset{=}{<} r$, $|y| \overset{=}{<} t$, say. (The symbol $|x|$ denotes the numerical value of x.)

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* This proof is taken from notes of a course on differential equations given by Professor David Hilbert at Göttingen.

† This is by far the most important case, since in the higher mathematics such functions are almost exclusively dealt with, and in applied mathematics they are universally used for approximations.

It is to be shown that there is a *convergent series*

$$y = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots \tag{2}$$

which *identically* satisfies

$$y' = f(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + \cdots + a_i x^m y^n + \cdots; \tag{3}$$

and which also satisfies a given initial condition, say, that $y = y_0$ when $x = x_0$.*

That $y = 0$ when $x = 0$ may be taken for the initial condition without any loss of generality. For, on substituting $x_1 + x_0$ for x and $y_1 + y_0$ for y in (1), it becomes

$$y_1' = \phi(x_1, y_1); \tag{4}$$

and it is evident that for

$$y_1 = \alpha_0' + \alpha_1' x + \alpha_2' x^2 + \cdots$$

to identically satisfy (4) and the initial condition that $y_1 = 0$ when $x_1 = 0$, is the same thing as for (2) to satisfy (3) and the initial condition that $y = y_0$ when $x = x_0$. Hence the initial condition may be taken in this form at the beginning; and for this it is both necessary and sufficient that α_0 in (2) be zero.

It will now be shown

(*a*) that there is *one* and *only one* series,

$$y = \alpha_1 x + \alpha_2 x^2 + \cdots, \tag{5}$$

which satisfies (3) identically; and

(*b*) that within certain limits for x this series is convergent.

On transforming the series in (3), which has been supposed convergent for $|x| \leqq r$, $|y| \leqq t$, by putting $x = rx_1$, $y = ty_1$, equation (3) takes the form

$$y' = f(rx_1, ty_1) = a_0' + a_1' x_1 + a_2' y_1 + a_3' x_1^2 + a_4' x_1 y_1 + \cdots.$$

The second member of this equation is a convergent series, and converges when $x_1 = y_1 = 1$; and, therefore, $a_0' + a_1' + a_2' + \cdots$ converges. This shows that the absolute value of each a' is not larger than a certain finite quantity A, say. The substitution just made for x and y does not make any essential change in the problem, and hence it might have been assumed at first that the a's of (3) were each not greater than A. In what follows the a's are accordingly regarded as not greater than A.

---

* If an initial condition be not made, then an infinite number of series can be found which will satisfy (3).

If (5) satisfies (3), the value of y and y' derived from (5), when substituted in (3), must make the latter an identity; and, therefore,

$$\begin{aligned}&\alpha_1 + 2\,\alpha_2 x + 3\,\alpha_3 x^2 + \cdots \\ &\quad = a_0 + a_1 x + a_2(\alpha_1 x + \alpha_2 x^2 + \cdots) + a_3 x^2 + a_4 x(\alpha_1 x + \alpha_2 x^2 + \cdots) \\ &\qquad + a_5(\alpha_1 x + \alpha_2 x^2 + \cdots)^2 + \cdots\end{aligned}$$

is an identical equation. Hence

$\alpha_1 = a_0$; $2\,\alpha_2 = a_1 + a_2\alpha_1$, whence $\alpha_2 = \dfrac{a_1 + a_2 a_0}{2}$; $3\,\alpha_3 = a_3 + a_2\alpha_2 + a_4\alpha_1$, whence $\alpha_3 = a_3 + \dfrac{a_2}{2}(a_1 + a_2 a_0) + a_4 a_0$; and similarly for α_4, α_5, $\cdots$. It is evident that all the α's can be determined as rational integral functions of the a's; and it is also to be noticed that all the numerical coefficients in the expressions for the α's are *positive;* and, therefore, *the α's will not be diminished if each of the a's is replaced by A.*

From the method of derivation it is evident that (5) with the α's determined as above identically satisfies (3). It has still to be determined whether this series is convergent.

On replacing each of the a's in (3) by A, a quantity not less than any one of the a's, there results

$$y' = A(1 + x + y + x^2 + xy + y^2 + x^3 + x^2 y + \cdots). \tag{6}$$

The integral of this equation is found by replacing each of the a's that occur in the expressions for the α's of (5) by A. None of these latter coefficients are diminished by changing each of the a's to A, as pointed out above; hence, if the integral of (6) is convergent, the integral of (3) is also.

Now solve (6) directly. On factoring the second member, the equation becomes

$$\begin{aligned}y' &= A(1 + x + x^2 + \cdots)(1 + y + y^2 + \cdots), \\ &= A\frac{1}{1-x}\cdot\frac{1}{1-y}.\end{aligned}$$

Therefore, $$(1-y)dy = A\frac{dx}{1-x};$$

whence, on integration, $y - \frac{1}{2}y^2 = -A\log(1-x) + c_1$.

Therefore, $$y = 1 \pm [2\,A\log(1-x) + c + 1]^{\frac{1}{2}}.$$

Here c must be determined, so that the initial condition be satisfied, namely, that $y = 0$ when $x = 0$; therefore

$$0 = 1 \pm \sqrt{c+1}.$$

Hence the square root must have the minus sign, and c must be zero.

Therefore,

$$y = 1 - [1 + 2A\log(1-x)]^{\frac{1}{2}} = 1 - \left[1 - 2A\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)\right]^{\frac{1}{2}}. \quad (7)$$

The series $x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ converges for $|x| < 1$; hence the square root of $1 - 2A\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$ converges for $|x| < 1$; and hence the value of y in (7) is finite; and, therefore, the value of y in (5) is finite for x within certain limits.

Note A showed that an equation of order n can be replaced by a system of n simultaneous equations of the first order, each containing an unknown function to be found. In the case of the equation of order n, the proof of the existence of integrals is made for this equivalent system instead of for the single equation of the nth order; the proof can be carried through in much the same way.*

The method of proof given above is known as "the Power-Series method."

Historical Note.† — Augustin Louis Cauchy (1789–1857) of Paris, who was one of the leaders in insisting on rigorous demonstrations in mathematical analysis, gave the two first proofs of the existence theorem for ordinary differential equations. The first proof was given for real variables in 1823 in his lectures at the Polytechnic School in Paris; the second was given in 1835 for complex variables in a lithographed memoir. He was also the first who proved the existence of integrals of a partial differential equation. The first of the two proofs was published in Moigno's Calculus in 1844; this may be called "the method of difference equations"; it has been developed and simplified by Gilbert in France and Lipschitz in Germany. In his second method Cauchy employed what he called "the Calculus of limits." This method has been developed by Briot and Bouquet, and Méray in France, and Weierstrass (1815–1897) in Germany. (The proof given above follows Weierstrass' exposition of Cauchy's second proof.) A new proof, that by "the method of successive approximations," was given by Émile Picard of Paris in 1890.‡

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* Leo Koenigsberger, *Theorie der Differentialgleichungen* (Leipzig, 1889), p. 27.

† For many historical notes and references relating to the existence theorem see Mansion, *Theorie der partiellen Differentialgleichungen*, pp. 26–29.

‡ For an English translation of this proof made by Professor T. S. Fiske, see *Bulletin N. Y. Math. Soc.*, Vol. I. (1891–1892), pp. 12–16.

In the *Traité d'Analyse* of É. Picard, t. II., pp. 291–318, will be found, besides the author's own proof just mentioned, Cauchy's first and second proofs, the latter as modified by Briot and Bouquet; and Madame Kowalevsky's proof* of the existence of integrals of a system of partial differential equations. (A knowledge of the theory of functions of a complex variable is necessary for the reading of some of these proofs.)

† NOTE C.

[This Note is supplementary to Art. 3.]

The complete solution of a differential equation of the nth order contains n arbitrary independent constants.

Let y', y'', $\cdots$ denote the first, second, $\cdots$ derivatives of y with respect to x, and $y(0)$, $y'(0)$, $y''(0)$, $\cdots$ denote the values of y, y', y'', $\cdots$ when $x = 0$. First, let an equation of the first order be considered; and suppose that the solution of

$$F(y', y, x) = 0, \tag{1}$$

when expanded in ascending powers of x is

$$y = c + c_1x + c_2x^2 + \cdots. \tag{2}$$

Note B shows that the solution can be thus expressed.

But $y(x) = y(0) + y'(0)\,x + \frac{1}{\underline{|2}}\,y''(0)x^2 + \cdots$ (Maclaurin's Theorem); (3)

and therefore $c = y(0)$, $c_1 = y'(0)$, $c_2 = \frac{1}{\underline{|2}}\,y''(0)$, $\cdots$.

Now $c = y(0)$ cannot be expressed in terms of anything known or determinable. However, $c_1 = y'(0)$ can be determined, for $F(y', y, x) = 0$ holds true for all values of x, and hence for $x = 0$; therefore $F\{y'(0), y(0), 0\} = 0$, that is $F(c_1, c, 0) = 0$. This determines c_1 in terms of c.

Equation (1) may be solved for y', thus,

$$y' = f(y, x); \tag{4}$$

then, on differentiation,

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* Crelle, Vol. 80. (Memoir dated 1874.) Madame Sophie de Kowalevsky (1853–1891) was professor of higher mathematics at Stockholm (1884–1891), and received the Bordin prize of the French Academy in 1888.

† For this Note, I am indebted to notes of lectures by Professor Hilbert at Göttingen.

$$y'' = \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial x}; \text{ therefore } y''(0) = \left[\frac{\partial f}{\partial y}y'(0) + \frac{\partial f}{\partial x}\right]_{x=0}.$$

This determines c_2 in terms of c and c_1, and from this c_2 can be found in terms of c alone. Another differentiation and the substitution of $x = 0$ in the result will give an equation by means of which $y'''(0)$, and thus c_3 also, can be expressed in terms of c; and similarly for the constants c_4, c_5, $\cdots$. Therefore all the constants except c are determined; that is, the differential equation of the first order has one arbitrary constant in its general solution.

In the next place, let an equation of the second order be considered. Put the equation $\phi(y'', y', y, x) = 0$ into the form

$$y'' = f(y', y, x); \tag{5}$$

and suppose that the solution is

$$y = c + c_1x + c_2x^2 + \cdots.$$

Determination of the values of c, c_1, c_2, $\cdots$, as before, gives $c = y(0)$, $c_1 = y'(0)$, $c_2 = \frac{1}{\lfloor\underline{2}}y''(0)$, $\cdots$. But, from the given equation,

$$y''(0) = f\{y'(0),\ y(0),\ 0\};$$

and this determines c_2 in terms of c and c_1. On differentiating (5) and putting $x = 0$, there is obtained

$$y'''(0 = \left[\frac{\partial f}{\partial y'}y'' + \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial x}\right]_{x=0} = F(c,\ c_1,\ c_2) = F\{c,\ c_1,\ \tfrac{1}{2}f(c_1, c, 0)\};$$

and hence c_3 is found in terms of c and c_1. By proceeding in this way, the values of all the other coefficients can be obtained in terms of c and c_1; but it will not be possible to obtain any information about c and c_1. The solution of (5) will therefore contain two arbitrary constants.

The proof of the theorem for equations of higher orders is made in exactly the same way as has just been used in the case of equations of the first and second orders.

NOTE D.

[This Note is supplementary to Art. 4.]

Criterion for the Independence of Constants of Integration.

In Art. 4 an example has been given of an integral in which there are apparently two constants of integration, but in reality these two are equivalent to only one. The question thus arises, how is it to be determined whether the constants of integration are really independent?

In the case of a solution of an equation of the second order having the form $y = \phi(x, c_1, c_2)$, the criterion that the constants c_1, c_2 be independent, by which it is meant that this solution be not reducible to the form $y = \psi\{x, f(c_1, c_2)\}$, in which there is really only one arbitrary constant $f(c_1, c_2)$, is that the determinant

$$\begin{vmatrix} \dfrac{\partial\phi}{\partial c_1} & \dfrac{\partial\phi}{\partial c_2} \\ \dfrac{\partial^2\phi}{\partial c_1\,\partial x} & \dfrac{\partial^2\phi}{\partial c_2\,\partial x} \end{vmatrix}$$

be not equal to zero.

For, suppose that $\phi(x, c_1, c_2)$ can be put in the form $\psi\{x, f(c_1, c_2)\}$. On forming and expanding the above determinant, there results

$$\frac{\partial\psi}{\partial f}\frac{\partial f}{\partial c_1}\cdot\frac{\partial^2\psi}{\partial x\,\partial f}\frac{\partial f}{\partial c_2} - \frac{\partial^2\psi}{\partial x\,\partial f}\frac{\partial f}{\partial c_1}\cdot\frac{\partial\psi}{\partial f}\cdot\frac{\partial f}{\partial c_2},$$

which is identically zero.

* Conversely, if the determinant be identically zero, then $\phi(x, c_1, c_2)$ must be of the form $\psi\{x, f(c_1, c_2)\}$; that is, $\phi(x, c_1, c_2)$ will not vary, no matter how c_1 and c_2 are varied, so long as $f(c_1, c_2)$ is assigned some particular constant value.

On writing p, q, for $\dfrac{\partial\phi}{\partial c_1}$, $\dfrac{\partial\phi}{\partial c_2}$, the condition that the determinant be zero takes the form $p\dfrac{\partial q}{\partial x} - q\dfrac{\partial p}{\partial x} = 0$, whence on integration $\dfrac{p}{q} =$ a constant; that is, $\dfrac{p}{q}$ is independent of x, and hence can only involve c_1 and c_2. Take $\dfrac{p}{q} = \dfrac{L}{M}$, where L, M, are functions of c_1, c_2. Hence $M\dfrac{\partial\phi}{\partial c_1} = L\dfrac{\partial\phi}{\partial c_2}$. Now differentiation of $\phi(x, c_1, c_2)$ gives

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial c_1}dc_1 + \frac{\partial\phi}{\partial c_2}dc_2 = \frac{\partial\phi}{\partial x}dx + (Ldc_1 + Mdc_2)\,\frac{1}{L}\frac{\partial\phi}{\partial c_1}.$$

But $Ldc_1 + Mdc_2$ has an integrating factor μ, such that $\mu Ldc_1 + \mu Mdc_2$ is a complete differential of the form $df(c_1, c_2)$.

Therefore $d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{1}{\mu L}\dfrac{\partial\phi}{\partial c_1}\cdot df(c_1, c_2)$; hence $\phi(x, c_1, c_2)$ will not vary, no matter how c_1, c_2 are varied, provided only that they satisfy the condition $f(c_1, c_2) =$ a constant.

Hence the necessary and sufficient condition that c_1, c_2 be really independent in $\phi(x, c_1, c_2)$ is that the above determinant be not equal to zero.

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* This part of the proof is due to Professor McMahon of Cornell University.

More generally, the criterion that the n parameters $c_1, c_2, \cdots, c_n$ in $f(x, c_1, c_2, \cdots, c_n)$ be independent is that the determinant

$$\begin{vmatrix} \frac{\partial f}{\partial c_1} & \frac{\partial f}{\partial c_2} & \cdots \frac{\partial f}{\partial c_n} \\ \frac{\partial^2 f}{\partial c_1 \partial x} & \frac{\partial^2 f}{\partial c_2 \partial x} & \cdots \frac{\partial^2 f}{\partial c_n \partial x} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ \frac{\partial^n f}{\partial c_1 \partial x^{n-1}} & & \cdots \frac{\partial^n f}{\partial c_n \partial x^{n-1}} \end{vmatrix}$$

be not equal to zero. This follows from the theorem proved in Note F.

NOTE E.

[This Note is supplementary to Art. 12.]

*** Proof that $Pdx + Qdy$ is an exact differential when $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.**

Let $$\int Pdx = V, \text{ then } \frac{\partial V}{\partial x} = P; \ \frac{\partial^2 V}{\partial x \partial y} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x};$$

therefore $$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right).$$

Hence $Q = \frac{\partial V}{\partial y} + \phi'(y)$, where $\phi'(y)$ is some function of y. Therefore

$$Pdx + Qdy = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \phi'(y)dy$$
$$= d[V + \phi(y)], \text{ an exact differential.}$$

NOTE F.

[This Note is supplementary to Art. 49.]

On the criterion that n integrals $y_1, y_2, \cdots, y_n$ of the linear differential equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0 \tag{1}$$

be linearly independent.

---

* I am indebted for this proof to Professor McMahon, of Cornell.

Before proceeding to establish the criterion, it may be remarked that if there be a linear relation

$$a_1y_1 + a_2y_2 + \cdots = 0, \tag{2}$$

where $a_1, a_2, \cdots$ are constants, existing between all or any of the integrals $y_1, y_2, \cdots, y_n$, then the integral $y = c_1y_1 + c_2y_2 + \cdots + c^ny_n$, in virtue of (2), may be written

$$y = \left(c_2 - c_1\frac{a_2}{a_1}\right)y_2 + \left(c_3 - c_1\frac{a_3}{a_1}\right)y_3 + \cdots + \left(c_n - c_1\frac{a_n}{a_1}\right)y_n.$$

This expression does not really contain more than $n - 1$ arbitrary constants, and therefore is not the general integral.

Form the determinant

$$\begin{vmatrix} y_1 & y_2 & \cdots y_n \\ y_1' & y_2' & \cdots y_n' \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots y_n^{(n-1)} \end{vmatrix},$$

where the elements of each row below the first are the derivatives of the corresponding elements in the row above them. This determinant is known as the *functional determinant* of $y_1, y_2, \cdots, y_n$, and will be denoted by R. The necessary and sufficient condition for the linear independence of $y_1, y_2, \cdots, y_n$ is that R be not equal to zero.

Suppose that this condition holds in the case of $n - 1$ functions, then it holds for n functions.

If there be a relation such as (2) between the functions $y_1, y_2, \cdots, y_n$, then the elements of one of the columns of R are formed from several other columns by adding the same multiples of the corresponding elements of these other columns; and, consequently, R will be identically equal to zero.

Conversely, if $R = 0$, there will be a linear relation of the form (2) between the functions $y, y_1, \cdots, y_n$. Since $R = 0$, the determinant must be reducible to a form wherein all the elements of one column are zero; that is, there must be certain multipliers $\lambda_1, \lambda_2, \cdots, \lambda_n$, such that

$$\left.\begin{array}{l} \lambda_1y_1 + \lambda_2y_2 + \cdots + \lambda_ny_n = 0 \\ \lambda_1y_1' + \lambda_2y_2' + \cdots + \lambda_ny_n' = 0 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \lambda_1y_1^{(n-1)} + \lambda_2y_1^{(n-1)} + \cdots + \lambda_ny_n^{(n-1)} = 0 \end{array}\right\} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \tag{3}$$

Differentiation of each of these equations and subtraction of the one next following gives

$$\left.\begin{array}{l}\lambda_1'y_1+\lambda_2'y_2+\cdots+\lambda_n'y_n=0\\ \lambda_1'y_1'+\lambda_2'y_2'+\cdots+\lambda_n'y_n'=0\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \lambda_1'y_1^{(n-2)}+\lambda_2'y_2^{(n-2)}+\cdots+\lambda_n'y_n^{(n-2)}=0\end{array}\right\}\quad\cdot\;\cdot\;\cdot\;\cdot\;\cdot\quad(4)$$

If one of the determinants $\begin{vmatrix} y_1 & y_2 & \cdots y_n \\ y_1' & y_2' & \cdots y_n' \\ \cdot & \cdot & \cdot \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots y_n^{(n-2)} \end{vmatrix}$ vanishes, say the one formed by omitting the rth column, then by hypothesis there is a relation

$$c_1y_1+c_2y_2+\cdots+c_{r-1}y_{r-1}+c_{r+1}y_{r+1}+\cdots+c_ny_n\equiv 0.$$

But if no one of these determinants vanishes, then it follows from (4) and the first $(n-1)$ equations in (3) that

$$\frac{\lambda_1'}{\lambda_1}=\frac{\lambda_2'}{\lambda_2}=\cdots=\frac{\lambda_n'}{\lambda_n}.$$

Suppose that each of these fractions is equal to ρ, say. It follows from integration that $\lambda_1=a_1e^{\int\rho dx}$, $\lambda_2=a_2e^{\int\rho dx}$, $\cdots$, $\lambda_n=a_ne^{\int\rho dx}$, a_1, a_2, $\cdots$, a_n being the constants of integration. On substituting these values in the first of equations (3) and dividing by the common factor $e^{\int\rho dx}$, there appears the relation

$$a_1y_1+a_2y_2+\cdots+a_ny_n=0,$$

which is thus a consequence of R being equal to zero. Hence, if the criterion holds for $n-1$ functions, it holds also for n. But it can be shown as in Note D that the criterion holds for 2 functions; hence it holds for 3, hence for 4, and so on for any number.

Therefore, the necessary and sufficient condition that y_1, y_2, $\cdots$, y_n form a system of linearly independent integrals, or a *fundamental system of integrals*, as it is sometimes called, is that the determinant R do not vanish identically.

NOTE G.

The relations between the coefficients of a linear differential equation and its integrals.

Let y_1, y_2, $\cdots$, y_n be n linearly independent functions of x. It is required to form the differential equation which has these functions for its integrals; in other words, to form the equation which has

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n \tag{1}$$

for its general solution. The differential equation is formed by eliminating $c_1, c_2, \cdots, c_n$ from the given integral by the method shown in Art. 3. By differentiating n times there is obtained the set of $n+1$ equations,

$$\begin{aligned} y &= c_1y_1 + c_2y_2 + \cdots + c_ny_n \\ y' &= c_1y_1' + c_1y_1' + \cdots + c_ny_n' \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ y_1^{(n)} &= c_1y_1^{(n)} + c_2y_2^{(n)} + \cdots + c_ny_n^{(n)}. \end{aligned}$$

From this the eliminant of the c's is found to be

$$\begin{vmatrix} y & y_1 & y_2 & \cdots y_n \\ y' & y_1' & y_2' & \cdots y_n' \\ \cdot & \cdot & \cdot & \cdot \quad \cdot \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \cdots y_n^{(n)} \end{vmatrix} = 0, \tag{2}$$

the differential equation required.

Now suppose that the differential equation having the integrals $y_1, y_2, \cdots, y_n$ is in the form

$$\frac{d^ny}{dx_n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n(y) = 0. \tag{3}$$

On denoting the minors of $y, y', \cdots, y^{(n)}$ in (2) by $Y, Y_1, \cdots, Y_n$, respectively, (2) on expansion becomes

$$Y_n\frac{d^ny}{dx^n} - Y_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + (-1)^nYy = 0. \tag{4}$$

Comparison of (3) and (4) shows that

$$P_1 = -\frac{Y_{n-1}}{Y_n},\ P_2 = \frac{Y_{n-2}}{Y_n}, \cdots, P_n = (-1)^n\frac{Y}{Y_n}.$$

It will be observed that Y_n is the determinant R of Note F. *In particular, since differentiation will show that $Y_{n-1} = \frac{dY_n}{dx}$, $P_1 = -\frac{1}{Y_n}\frac{dY_n}{dx}$; and hence $Y_n = e^{-\int P_1dx}$.

NOTE H.

[This Note is supplementary to Art. 102.]

On the criterion of integrability of $Pdx + Qdy + Rdz = 0$.

It has been shown in Art. 102 that the *necessary* condition for the existence of an integral of

$$Pdx + Qdy + Rdz = 0 \tag{1}$$

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* This deduction is due to Joseph Liouville (1809–1882), professor at the Collége de France.

is that the coefficients P, Q, R, satisfy the relation

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \tag{2}$$

It will now be proved that this condition is also *sufficient*, by showing that an integral of (1) can be found when relation (2) holds.

Substitution shows that, if relation (2) holds for the coefficients of (1), a similar relation holds for the coefficients of

$$\mu P dx+\mu Q dy+\mu R dz=0, \tag{3}$$

where μ is any function of x, y, z. If $Pdx+Qdy$ is not an exact differential with respect to x and y, an integrating factor μ can be found for it, and (3) can then be taken as the equation to be considered. Hence there is no loss of generality in regarding $Pdx+Qdy$ as an exact differential.

On assuming then that

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x},$$

and that

$$V=\int(Pdx+Qdy),$$

it follows that

$$P=\frac{\partial V}{\partial x}, \qquad Q=\frac{\partial V}{\partial y},$$

$$\frac{\partial P}{\partial z}=\frac{\partial^2 V}{\partial z\partial x}, \quad \frac{\partial Q}{\partial z}=\frac{\partial^2 V}{\partial z\partial y}.$$

Hence, from (2),

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z\partial y}-\frac{\partial R}{\partial y}\right)+\frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x}-\frac{\partial^2 V}{\partial z\partial x}\right)=0.$$

This may be written

$$\frac{\partial V}{\partial x}\cdot\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z}-R\right)-\frac{\partial V}{\partial y}\cdot\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z}-R\right)=0,$$

or

$$\begin{vmatrix} \dfrac{\partial V}{\partial x}, & \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial z}-R\right) \\ \dfrac{\partial V}{\partial y}, & \dfrac{\partial}{\partial y}\left(\dfrac{\partial V}{\partial z}-R\right) \end{vmatrix}=0.$$

This equation shows that a relation independent of x and y exists between

$$V \text{ and } \frac{\partial V}{\partial z}-R.$$

Therefore, $\frac{\partial V}{\partial z} - R$ can be expressed as a function of z and V alone. Suppose that

$$\frac{\partial V}{\partial z} - R = \phi(z, V). \tag{4}$$

Since $Pdx + Qdy + Rdz \equiv \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz + \left(R - \frac{\partial V}{\partial z}\right)dz$, equation (1) may be written, on taking account of (4),

$$dV - \phi(z, V)\,dz = 0.$$

This is an equation in two variables. Its integration will lead to an equation of the form

$$F(V, z) = 0.$$

Hence (2)* is both the necessary and sufficient condition that (1) have an integral.

† NOTE I.

Modern Theories of Differential Equations. Invariants of Differential Equations.

The two modern theories of differential equations are:

(*a*) The theory based upon the theory of functions of a complex variable;

(*b*) The theory based upon Lie's theory of transformation groups.

The study of differential equations, until about forty years ago, was restricted to the derivation of rules and methods for obtaining solutions of the equation and expressing these solutions in terms of known functions. Even at the beginning of the present century,‡ however,

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[*] Of course this criterion is included in the criterion for the general case of p variables, the deduction and proof of which is to be found in Forsyth, *Theory of Differential Equations*, Part I., pp. 4–12. (See footnote, p. 138.) See Serret, *Calcul Intégral* (edition 1886), Arts. 785–786.

[†] Two historical articles that the student would do well to consult are: T. Craig, "Some of the developments in the theory of ordinary differential equations between 1878 and 1893," *Bulletin of N. Y. Math. Soc.*, Vol. II. (1892–1893), pp. 119–134; D. E. Smith, "History of Modern Mathematics" (Merriman and Woodward, *Higher Mathematics*, Chap. XI.), Art. 11. Also see F. Cajori, *History of Mathematics*, pp. 341–347.

[‡] "Gauss in 1799 showed that the differential equation meets its limitations very soon, unless complex numbers are introduced."

mathematicians saw that any marked advance in this direction was impossible without the aid of new conceptions and new methods. But it was not until a comparatively recent date, that wider regions were discovered and began to be explored.

"A new era began with the foundation of what is now called function-theory by Cauchy, Riemann, and Weierstrass. The study and classification of functions according to their essential properties, as distinguished from the accidents of their analytical forms, soon led to a complete revolution in the theory of differential equations. It became evident that the real question raised by a differential equation is not whether a solution, assumed to exist, can be expressed by means of known functions, or integrals of known functions, but in the first place whether a given differential equation does really suffice for the definition of a function of the independent variable (or variables), and, if so, what are the characteristic properties of the function thus defined. Few things in the history of mathematics are more remarkable than the developments to which this change of view has given rise." *

The leading events in the early history of this new theory are: the publication of the memoir on the properties of functions defined by differential equations, by Briot and Bouquet in the *Journal de l'École Polytechnique* (Cahier 36) in 1856; the paper on the differential equation which satisfies the Gaussian series, by Riemann at Göttingen in 1857; and, perhaps, most important of all, the appearance of the memoirs of Fuchs on the theory of linear differential equations with variable coefficients, in *Crelle's Journal* (Vols. 66, 68) in 1866 and 1868. †

The only work in English which employs the function-theory method in discussing differential equations is that of Professor Craig. ‡

A knowledge of the theory of substitutions, as well as of function-theory, is required for reading some of the modern articles on differential equations.

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* See G. B. Mathews, a review in *Nature*, Vol. LII. (1895), p. 313.

† Albert Briot (1817–1882); Jean Claude Bouquet (1819–1885); Georg Friedrich Bernhard Riemann (1826–1866), the founder of a general theory of functions of a complex variable, and the inventor of the surfaces, known as "Riemann's surfaces"; Lazarus Fuchs (born 1835), professor at Berlin.

‡ T. Craig, *Treatise on Linear Differential Equations* (Vol. I., published in 1889). See Note J for the names of other works on the modern theories.

Professor Lie[*] of Leipzig has discovered, and since 1873 has developed, the theory of transformation groups. This theory bears a close analogy to Galois' theory of substitution groups which play so large a part in the treatment of algebraic equations. By means of Lie's theory it can be at once discovered whether or not a differential equation can be solved by quadratures.[†] An elementary work by Professor J. M. Page on differential equations treated from the standpoint of Lie's theory has been published.[‡]

The theory of *invariants* of linear differential equations is one of the later developments in the study of differential equations. While it plays a very important part in both of the modern theories referred to above, yet, to some extent, it can be studied without a knowledge of these theories.[§] It has been found that differential equations, like algebraic equations, have invariants. An invariant of a linear differential equation is a function of its coefficients and their derivatives, such that, when the dependent variable undergoes any linear transformation, and the independent variable any transformation whatsoever, this function is equal to the same function of the coefficients of the new equation multipled by a certain power of the derivative of the new independent variable with respect to the old.

The introduction of invariants into the study of differential equations is due to E. Laguerre of Paris.[||] Those who have made the most important investigations on these invariants are Halphen and Professor Forsyth. Their memoirs* are among the principal sources of information on the subject.

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[*] Sophus Lie was born in Norway and educated in Christiania. He has been Professor of Geometry at Leipzig since 1886. He has expounded his theory in the following works: *Theorie der Transformationgruppen*, Vols. I., II., III. (1888–1893); *Vorlesungen über continuierliche Gruppen* (1893). See p. 207 for his work on Differential Equations.

[†] For an elementary introduction to Lie's theory of transformation groups, and its application to differential equations, see articles by J. M. Page: "Transformation Groups," *Annals of Mathematics*, Vol. VIII., No. 4 (1894), pp. 117–133; "Transformation groups applied to ordinary differential equations," *Annals of Mathematics*, Vol. IX., No. 3 (1895), pp. 59–69. Also see J. M. Brooks, "Lie's Continuous Groups," a review in *Bull. Amer. Math. Soc.*, 2d Series, Vol. I., p. 241.

[‡] By The Macmillan Co.

[§] See Craig, *Linear Differential Equations*, pp. 19–22, 463–471; and the memoir of Forsyth referred to below.

[||] In his memoirs: "On linear differential equations of the third order," *Comptes Rendus*, Vol. 88 (1879), pp. 116–119; "On some invariants of linear differential equations," *Ibid.*, pp. 224–227.

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* G. H. Halphen (1844–1889) of the Polytechnic School in Paris. "Mémoire sur la réduction des équations différentielles lineaires aux formes intégrables," *Mémoires des Savants Étrangers*, Vol. 28 (1884), pp. 1–301. Chap. III., pp. 114–176, in this memoir treats of invariants.

A. R. Forsyth, "Invariants, Covariants, and Quotient-Derivatives associated with linear differential equations," *Phil. Trans. Roy. Soc.*, Vol. 179 (1888), A, pp. 377–489.

NOTE J.

[This Note is supplementary to Art. 53.]

The Symbol D.

Let D be a symbol which represents differentiation, with respect to x say, on the function immediately following it. In other words, let

$$Du \equiv \frac{d}{dx}(u), \text{ or } Du \equiv \frac{du}{dx}. \tag{1}$$

Then
$$D(Du) = \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d^2u}{dx^2}, \tag{2}$$

$$D\{D(Du)\} = \frac{d}{dx}\left(\frac{d^2u}{dx^2}\right) = \frac{d^3u}{dx^3}. \tag{3}$$

It is evident from definition (1) and the results (2), (3), that the result of the operation symbolized by D taken n times in succession will be $\frac{d^nu}{dx^n}$.

Also, let the operations which consist of the operation D repeated two, three, $\cdots$, n times in succession be denoted by D^2, D^3, $\cdots$, D^n. It should be noted that, according to this definition, D^nu represents $\frac{d^nu}{dx^n}$ and not $\left(\frac{du}{dx}\right)^n$. From this definition of D^n it follows that the operational symbol D is subject to the fundamental laws of algebra. For,

$$(D^r + D^n)u = \frac{d^ru}{dx^r} + \frac{d^nu}{dx^n} = \frac{d^nu}{dx^n} + \frac{d^ru}{dx^r} = (D^n + D^r)u\,;$$

$$D^r \cdot D^nu = \frac{d^r}{dx^r}\left(\frac{d^nu}{dx^n}\right) = \frac{d^{n+r}u}{dx^{n+r}} = \frac{d^n}{dx^n}\left(\frac{d^ru}{dx^r}\right) = D^n \cdot D^ru\,;$$

$$D^r \cdot D^nu = D^{n+r}u\,;$$

$$D^m(u+v) = \frac{d^m}{dx^m}(u+v) = \frac{d^mu}{dx^m} + \frac{d^mv}{dx^m} = D^mu + D^mv.$$

Since D represents an operation, it can only appear with integral exponents. Negative exponents will now be considered.

Suppose that
$$Du = v, \tag{4}$$
and let u be indicated by
$$u = D^{-1}v. \tag{5}$$

It is necessary to give a meaning to D^{-1}, and this meaning must not be inconsistent with the definition of D. On operating on each member of (5) with D,

$$Du = D \cdot D^{-1}v,$$

whence by (4), $D \cdot D^{-1}v = v.$

Therefore D^{-1} represents such an operation on any function that, if the operation represented by D be subsequently performed, the function is left unaltered. Hence the operation represented by D^{-1} is equivalent to an integration. It follows that the operation indicated by D^{-n} is equivalent to n successive integrations. The proof that the symbol D with negative exponents is subject to the laws of algebra, is similar to that used for D with positive exponents.

It has been seen that $D \cdot D^{-1}v = v.$

But $D^{-1} \cdot Dv = v + c,$

in which c is an arbitrary constant of integration. Therefore, in order that

$$D^m \cdot D^{-n}v \equiv D^{-n} \cdot D^m v,$$

it is necessary to omit the arbitrary constant that arises when the operation indicated by D^{-1} is performed.

NOTE K.

[This Note is supplementary to Art. 82.]

Integration in series.

The law for the exponents will be apparent on substituting x^m for y in the first member of the given equation. Suppose that the expression obtained by this substitution is

$$f_1(m)x^{m'} + f_2(m)x^{m''}. \tag{3}$$

In general (3) will contain more than two terms; in the case of the equations in Art. 66 it contains only one term. Under the supposition just made, the successive differences of the exponents of x in the series sought must evidently be $m'' - m'$. This common difference will be denoted by s. Solution (2) may now be written

$$y = A_0x^m + A_1x^{m+s} + \cdots + A_{r-1}x^{m+(r-1)s} + A_rx^{m+rs} + \cdots, \tag{4}$$

or simply $y = \sum_{r=0}^{r=\infty} A_rx^{m+rs}.$

Substitution of this series for y in (1) will give, in virtue of (3),

$$\left.\begin{aligned}&A_0 f_1(m)\,x^{m'} + A_0 f_2(m)\,x^{m'+s}\\&+ A_1 f_1(m+s)\,x^{m'+s} + A_1 f_2(m+s)\,x^{m'+2s}\\&+\cdots\\&+ A_{r-1} f_1[m+(r-1)s]\,x^{m'+(r-1)s}\\&+ A_{r-1} f_2[m+(r-1)s]\,x^{m'+rs}\\&+ A_r f_1(m+rs)\,x^{m'+rs}\\&+ A_r f_2(m+rs)\,x^{m'+(r+1)s}\\&+\cdots\end{aligned}\right\}=0. \tag{5}$$

Since equation (5) must be an identity, the coefficients of each power of x therein must be equal to zero; hence

$$f_1(m)=0 \tag{6}$$

and $$A_r f_1(m+rs) + A_{r-1} f_2[m+(r-1)s] = 0. \tag{7}$$

The roots of (6) give the initial exponents of series that will satisfy (1); and equation (7) shows that

$$A_r = -\frac{f_2[m+(r-1)s]}{f_1(m+rs)}A_{r-1},$$

which is the relation between successive coefficients. The difference between the exponents in (3) might have been taken, $m' - m''$ or $-s$; in this case, the resulting series would have had their powers in reverse order to those of (4); and the initial terms would have been found by solving $f_2(m)=0$.

In determining the initial power of x for an equation of the nth order, that coefficient in (3) which is of the nth degree in m must be put equal to zero, since there must be n independent series in the general solution. If both $f_1(m)$ and $f_2(m)$ are of the nth degree, two sets of series can be derived, one in ascending powers and the other in descending powers of x.

If the expression (3) have another term $f_3(m)x^{m'''}$, the terms of the series can be successively deduced, but the process will be much more tedious. This method can also be employed in the case of non-linear equations, but more than a very few terms can be calculated only with difficulty. The equations previously considered can of course be integrated in series; Ex. 3, Art. 82, illustrates this.

ANSWERS TO THE EXAMPLES.

ANSWERS TO THE EXAMPLES.

CHAPTER I.

$\left(p \text{ stands for } \dfrac{dy}{dx}.\right)$

Art. 3.

2. $(x^2 - 2y^2)p^2 - 4pxy - x^2 = 0.$

3. $(1 + p^2)^3 = r^2\left(\dfrac{d^2y}{dx^2}\right)^2.$

4. $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0.$

Page 11.

1. $\dfrac{dx}{\sqrt{1-x^2}} = \dfrac{dy}{\sqrt{1-y^2}}.$

2. $p\sqrt{1-x^2} = y.$

3. $\dfrac{d^3y}{dx^3} = 7\dfrac{dy}{dx} - 6y.$

4. $12p^2y = (8p^3 - 27)x.$

5. $y = px + p - p^3.$

6. $8ap^3 = 27y.$

7. $p^2(1 - x^2) + 1 = 0.$

8. $x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy = 0.$

9. $\dfrac{d^2y}{dx^2} + m^2y = 0.$

10. $2a\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 = 0.$

11. $y^2 = 2pxy + x^2.$

12. $\dfrac{d^3y}{dx^3} = 0.$

13. $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 = y\dfrac{dy}{dx}.$

14. $x^2\dfrac{d^2y}{dx^2} + 2y = 2x\dfrac{dy}{dx}.$

CHAPTER II.

Art. 8.

2. $y\sqrt{1-x^2} + x\sqrt{1-y^2} = c.$

3. $y = c(a + x)(1 - ay).$

4. $\tan y = c(1 - e^x)^3.$

Art. 9.

2. $xy^2 = c^2(x + 2y)$. **3.** $y = ce^{\frac{x^3}{3y^3}}$.

4. $c(2y^2 + 2xy - x^2)^{2\sqrt{3}} = \frac{(\sqrt{3} + 1)x + 2y}{(1 - \sqrt{3})x + 2y}$.

Art. 10.

1. $(y - x + 1)^2(y + x - 1)^5 = c$.

2. $c(y^2 - 5xy + x^2 + 11x + 4y - 17) = \left\{\frac{2y - (5 + \sqrt{21})x + 2(2 + \sqrt{21})}{2y - (5 - \sqrt{21})x + 2(2 - \sqrt{21})}\right\}^{\frac{1}{\sqrt{21}}}$.

Art. 13.

3. $a^2x - \frac{y^3}{3} - xy^2 - x^2y = c$. **4.** $ax^2 + bxy + cy^2 + gx + ey = k$.

5. $x^2y^2 + 4x^3y - 4xy^3 + y^3 - xe^y + e^{2x}y + x^4 = c$.

Art. 16.

3. $2a \log x + a \log y - y = c$. **4.** $x^2e^x + my^2 = cx^2$. **5.** $x^2 + \frac{e^x}{y} = c$.

Art. 17.

1. $\frac{x}{y} + \log \frac{y^3}{x^2} = c$. **3.** $2 \log x - \log y = \frac{1}{xy} + c$.

Art. 18.

1. $e^x(x^2 + y^2) = c$. **2.** $x^2 - y^2 = cx$. **3.** $x^3y^3 + x^2 = cy$.

4. $xy + y^2 + \frac{2x}{y^2} = c$.

Art. 19.

2. $5x^{-\frac{36}{13}}y^{\frac{24}{13}} - 12x^{-\frac{10}{13}}y^{-\frac{15}{13}} = c$. **3.** $6\sqrt{xy} - x^{-\frac{3}{2}}y^{\frac{3}{2}} = c$.

Art. 20.

2. $y = (x + c)e^{-x}$. **3.** $y = \tan x - 1 + ce^{-\tan x}$. **4.** $y = (e^x + c)(x + 1)^n$.

5. $3(x^2 + 1)y = 4x^3 + c$.

Art. 21.

3. $7y^{-\frac{1}{3}} = cx^{\frac{2}{3}} - 3x^3$. **4.** $y^{\frac{1}{2}} = c(1 - x^2)^{\frac{1}{4}} - \frac{1 - x^2}{3}$.

5. $y^3 = ax + cx\sqrt{1 - x^2}$.

Page 29.

1. $\dfrac{x+y}{a} = \tan\dfrac{y+c}{a}$.

2. $x^2 = 2cy + c^2$.

3. $y = \dfrac{x}{2}\left(ce^x - \dfrac{1}{ce^x}\right)$.

4. $\tan x \tan y = k$.

5. $\log\dfrac{x}{y} - \dfrac{y+x}{xy} = c$.

6. $y = x^2(1 + ce^{\frac{1}{x}})$.

7. $60\,y^3(x+1)^2 = 10\,x^6 + 24\,x^5 + 15\,x^4 + c$.

8. $x^2 - xy + y^2 + x - y = c$.

9. $y = ce^{-\frac{x}{\sqrt{1-x^2}}} + \dfrac{x}{\sqrt{1-x^2}}$.

10. $cx = e^{\frac{x}{y}}$.

11. $x^4 - y^4 + 2\,x^2y^2 - 2\,a^2x^2 - 2\,b^2y^2 = c$.

12. $y(x^2+1)^2 = \tan^{-1}x + c$.

13. $\log cy = \dfrac{x^3}{3\,y^3}$.

14. $y = ax + cx\sqrt{1-x^2}$.

15. $x^2 - y^2 - 1 = cx$.

16. $\log\sqrt{x^2+y^2} - m\tan^{-1}\dfrac{y}{x} = c$.

17. $r = ce^{m\theta}$.

18. $x + ye^{\frac{x}{y}} = c$.

19. $y^{-n+1} = ce^{(n-1)\sin x} + 2\sin x + \dfrac{2}{n-1}$.

20. $(x+1)e^y = 2\,x + c$.

21. $\dfrac{1}{y^2} = x^2 + 1 + ce^{x^2}$.

22. $\dfrac{ay^{n+2}}{n+2} = \dfrac{y^2}{x} + c$.

23. $y + 2\,y^3 + \dfrac{x^3}{3} - \dfrac{3}{2}\,x^2y^2 = c$.

24. $y^4 + 2\,a^2y^2 + x^4 - 2\,a^2x^2 + 2\,x^2y^2 = c$.

25. $\dfrac{1}{x} = 2 - y^2 + ce^{-\frac{1}{2}y^2}$.

26. $y^2 = ax^2 + c^2$.

27. $(x + \sqrt{a^2+x^2})y = a^2\log(x + \sqrt{a^2+x^2}) + c$.

28. $\log\sqrt{x^2+y^2} + \tan^{-1}\dfrac{y}{x} = c$.

29. $(4\,b^2+1)y^2 = 2\,a(\sin x + 2\,b\cos x) + ce^{-2bx}$.

30. $x^2 + y^2 = cy$.

31. $3\,y^2 - 2\,x^2e^{\frac{1}{x^3}} = cx^2$.

32. $c(y-b) = \dfrac{x}{1+bx}$.

33. $9\log(3\,y + 2\,x + \frac{22}{7}) = 14(3\,y - \frac{3}{2}\,x + c)$.

34. $x^2y^2 - 2\,xy\log cy = 1$.

35. $4x^2y = 5 + cx^{\frac{4}{7}}y^{\frac{12}{7}}$.

36. $cy = e^{\frac{y}{x}}$.

37. $x^ny = ax + c$.

38. $\frac{a}{2}\log\frac{x-y+a}{x-y-a} + y = c$.

CHAPTER III.

Art. 22.

3. $y = c,\ x + y = c,\ xy + x^2 + y^2 = c$.

4. $343(y+c)^3 = 27\,ax^7$.

5. $x^2 + 2y^2 = c,\ x^3 + y^2 = c$.

6. $y = 4x + c,\ y = 3x + c$.

Art. 24.

2. $x + c = \frac{a}{2}\{\log(1+p^2)^{-\frac{1}{2}}(p-1) - \tan^{-1}p\}$, with the given relation.

3. $\log(p-x) = \frac{x}{p-x} + c$, with the given relation.

4. $2y = cx^2 + \frac{a}{c}$.

Art. 25.

1. $y = c - [p^2 + 2p + 2\log(p-1)],\ x = c - [2p + 2\log(p-1)]$.

2. $y = c - a\log(p-1),\ x = c + a\log\frac{p}{p-1}$.

3. $y^2 = 2cx + c^2$.

Art. 26.

1. $x = \log p^2 + 6p + c$.

2. $y - c = \sqrt{x - x^2} - \tan^{-1}\sqrt{\frac{1-x}{x}}$.

3. $2y + c = a[p\sqrt{1+p^2} - \log(p + \sqrt{1+p^2})],\ x = a\sqrt{1+p^2}$.

4. $x + c = a\log(p + \sqrt{1+p^2}),\ y = a\sqrt{1+p^2}$.

Art. 27.

1. $y^2 + cyx^{\frac{2}{\sqrt{5}-1}} = c^2x^{\frac{4}{\sqrt{5}-1}}$.

2. $y^2 = 2cx + c^2$.

Art. 28.

3. $y = cx + \sin^{-1}c$.

4. $e^{2y} = ce^{2x} + c^2$.

5. $y^2 = cx^2 + 1 + c$

Page 38.

1. $\sin^{-1}\frac{y}{x} = \log cx$.

2. $y = c(x - b) + \frac{a}{c}$.

3. $(x^2 - y^2 + c)(x^2 - y^2 + cx^4) = 0.$ **5.** $(y - 6x + c)(y - 3x + c) = 0.$

4. $xy = c + c^2x.$ **6.** $ac^2 + c(2x - b) - y^2 = 0.$

7. $\theta + c = \int \frac{\phi(r^2)dr}{r\sqrt{r^2 - \phi^2(r^2)}}$, where $\theta = \tan^{-1}\frac{y}{x}$, and $r^2 = x^2 + y^2.$

8. $\tan^{-1}\frac{y}{x} + c = \text{vers}^{-1} 2a\sqrt{x^2 + y^2}.$

9. $\sin^{-1}\frac{y}{x} + \sin^{-1}\frac{1}{x} = c.$

10. $x^2 + y^2 - 4cx + 3c^2 = 0.$ (Put $x^2 - 3y^2 = v^2$.)

11. $x^2 + y^2 + 2c(x + y) + c^2 = 0.$

12. $y + \sqrt{y^2 + nx^2} = cx^{1 \pm \sqrt{\frac{n-1}{n}}}.$ **14.** $y^2 - cx^2 + \frac{ch^2}{c+1} = 0.$

13. $y^2 - b = (x + c)^2.$ **15.** $y(1 \pm \cos x) = c.$

16. $\left(y - \sin^{-1}\frac{x}{a} - c\right)\left(y - \cos^{-1}\frac{x}{a} - c\right)\left(y^{\frac{1}{2}} - x^{\frac{1}{2}} - c\right) = 0.$

17. $(y + c)^2 + (x - a)^2 = 1.$ **19.** $(y - cx^2)(y^2 + 3x^2 - c) = 0.$

18. $y = 2c\sqrt{x} + f(c^2).$ **20.** $y = c(x - c)^2.$

21. $(x^3 - 3y + c)(e^{\frac{x^2}{2}} + cy)(xy + cy + 1) = 0.$

22. $ax + c = \frac{3}{2}p^2 - mp + m^2 \log(p + m)$, with the given relation.

23. $e^y = ce^x + c^3.$ **25.** $(x + c)^2 + (y - b)^2 = 1.$

24. $\log \frac{y}{x + \sqrt{x^2 - y^2}} = c \pm y.$ **26.** $y = cx + \frac{m}{c}.$

27. $y^2 = cx + \frac{1}{8}c^3.$

CHAPTER IV.

Art. 33.

2. $y = cx + c^2,\ x^2 + 4y = 0.$ **4.** $x^2(y^2 - 4x^3) = 0.$

3. $(y + x - c)^2 = 4xy,\ xy = 0.$ **5.** $(x - y + c)^3 = a(x + y)^2,\ x + y = 0.$

Page 49.

1. $2y = cx^2 + \frac{a}{c},\ y^2 = ax^2.$

2. $a^3x + cxy + c^2 = 0$, singular solution is $x(xy^2 - 4a^3) = 0.$ $x = 0$ is also a tac-locus.

3. $(y - cx)^2 = m^2 + c^2,\ y^2 + m^2x^2 = m^2.$

4. $y = cx + \sqrt{b^2 + a^2c^2}$, $b^2x^2 + a^2y^2 = a^2b^2$.

5. $y = cx - c^2$, $x^2 = 4\,y$.

6. $y^2 = 4\,a(x-b)$; $1 + 4\,x^2y = 0$, $x = 0$ is a tac-locus; $y(27\,y - 4\,x^3) = 0$; $y^2 = 4\,mx$.

7. $(y+c)^2 = x^3$; $x = 0$ is a cusp locus; there is no singular solution.

8. $(y+c)^2 = x(x-1)(x-2)$; singular solutions are $x = 0$, $x = 1$, $x = 2$; $x = 1 \pm \dfrac{1}{\sqrt{3}}$ are tac-loci. The curve when $c = 0$ consists of an oval cutting the axis of x at the origin and at $x = 1$, and a curve resembling a parabola in shape, having its vertex at the point for which $x = 2$.

9. $x\{x^3 - c(y+c)^2\} = 0$; singular solution is $4\,y^3 + 27\,x^3 = 0$; $x = 0$ is a part of the general solution, and is the cusp locus for one part of the general solution and the envelope locus for the other part.

10. $y = cx + \sqrt{a^2c^2 + b^2}$, $b^2x^2 + a^2y^2 = a^2b^2$.

11. $x^2 + y^2 - c(x^2 - y^2) - 1 + c^2 = 0$; that is, $\dfrac{x^2}{1+c} + \dfrac{y^2}{1-c} = 1$. Singular solution is $x^4 - 2\,x^2y^2 + y^4 - 4\,x^2 - 4\,y^2 + 4 = 0$; that is, $(x + y + \sqrt{2})(x + y - \sqrt{2})(x - y + \sqrt{2})(x - y - \sqrt{2}) = 0$. The general solution is the system of conics touching these four lines.

CHAPTER V.

Art. 43.

3. $\sqrt{ny} = x + c$.

4. $y + c = \sqrt{ax - x^2} + \dfrac{a}{2}\,\text{vers}^{-1}\,\dfrac{2\,x}{a}$.

5. $r = c - \kappa\cos\theta$; when $c = \kappa$, the cardioid $r = \kappa(1 - \cos\theta)$.

6. $cr = e^{\frac{\theta}{\kappa}}$.

Art. 47.

3. The ellipses $2\,x^2 + y^2 = c^2$.

4. $y^{\frac{4}{3}} - x^{\frac{4}{3}} = c^{\frac{4}{3}}$.

5. The confocal and coaxal parabolas $r = \dfrac{2\,c}{1 - \cos\theta}$.

6. $\sec 5\,\theta + \tan 5\,\theta = ce^{\frac{25}{r}}$.

7. $r = ce^{-\phi\cot\theta}$, $r = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}\dfrac{y}{x}$.

Art. 48.

2. $s = \frac{1}{2}\,at^2 + v_0t + s_0$.

3. $s = \frac{1}{2}\,gt^2$.

Page 60.

1. $y = ce^{\kappa x}$.

2. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

3. $3\,y^2 = 2\,\kappa(x^3 + c)$.

4. $s = \kappa \sin \phi$, the intrinsic equation of a cycloid referred to its vertex, the radius of the generating circle being $\frac{1}{4}\kappa$.

5. The lines $r \sin(\theta + \alpha) = \frac{1}{\kappa}$, and their envelope the circle $r = \frac{1}{\kappa}$.

6. The parallel lines $(m \sin \alpha - n \cos \alpha)x - (m \cos \alpha + n \sin \alpha)y = c$.

7. $x^2 + 2y^2 = c$.

8. The system of circles passing through the given point and having their centres in the given line.

9. $x^2 + y^2 = 2a^2 \log x + c$. 10. $x^2 - y^2 = \pm c^2$.

11. $r^n = c^n \sin n\theta$; $r^2 = c^2 \sin 2\theta$, a series of lemniscates having their axis at an angle of 45° to that of the given system.

12. $r^2 - \kappa^2 = cr \operatorname{cosec} \theta$. 13. $r = e^{\sqrt{c^2 - \theta^2}}$.

14. Parabola $(y - x)^2 - 2a(y + x) + a^2 = 0$.

15. $x^2 + y^2 = a^2$. 16. $x^2 + y^2 = 2cx$.

17. $\log \frac{c^2}{x^3}(y + \sqrt{y^2 - x^2}) = \frac{y}{x^2}(y + \sqrt{y^2 - x^2})$.

18. $x^2 e^{\frac{y^2}{x^2}} = c$.

19. $r^n = c \sin n\theta$; $r = c \sin \theta$; $r = c(1 - \cos \theta)$.

20. $r = c(1 - \cos \theta)$.

21. $(y - x)^{\sqrt{2}} = c \frac{\sqrt{y} + \sqrt{x}}{\sqrt{y} - \sqrt{x}}$.

22. $re^{\frac{\sqrt{1-\kappa^2}}{\kappa}\theta} = c$.

23. $r = \frac{a(1 - e^2)}{1 - e \sin(\theta + c)}$.

24. $r = c(1 + \cos \theta)$.

27. The conics that have the fixed points for foci.

28. The conics that have the fixed points for foci.

29. The ellipse $\kappa^2 x^2 + a^2 y^2 = a^2 \kappa^2$. 30. The hyperbola $2xy = a^2$.

31. The parabola $a^2 x^2 = \kappa^2(2ay + \kappa^2)$.

32. The catenary $y = a \cosh \frac{x}{a}$. (See Johnson, *Diff. Eq.*, Art. 70.)

33. $4ay + c = 2ax\sqrt{4a^2x^2 - 1} - \log(2ax + \sqrt{4a^2x^2 - 1})$.

34. (a) $i = ce^{-\frac{R}{L}t} + \frac{1}{L}e^{-\frac{R}{L}t} \int e^{\frac{R}{L}t} f(t)\,dt$.

(b) $i = ce^{-\frac{R}{L}t}$. If $i = I$ when $t = 0$, $i = Ie^{-\frac{R}{L}t}$.

(c) $i = ce^{-\frac{R}{L}t} + \frac{E}{R}$. If $i = 0$ when $t = 0$, $i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$.

(d) $i = ce^{-\frac{R}{L}t} + \frac{E}{R^2 + L^2\omega^2}(R \sin \omega t - L\omega \cos \omega t).$

(e) $i = ce^{-\frac{R}{L}t} + \frac{E_1}{R^2 + L^2\omega^2}(R \sin \omega t - L\omega \cos \omega t)$

$+ \frac{E_2}{R^2 + L^2b^2\omega^2}\{R \sin (b\omega t + \theta) - Lb\omega \cos (b\omega t + \theta)\}.$

35. (a) $i = \frac{e^{-\frac{t}{RC}}}{R}\left[\int e^{\frac{t}{RC}} f'(t)dt + cR\right].$

(b) $i = ce^{-\frac{t}{RC}}.$ (c) $i = ce^{-\frac{t}{RC}}.$

(d) $i = ce^{-\frac{t}{RC}} + \frac{CE\omega}{1 + R^2C^2\omega^2}(\cos \omega t + RC\omega \sin \omega t).$

36. (a) $q = \frac{e^{-\frac{t}{RC}}}{R}\int e^{\frac{t}{RC}} f(t)dt + ce^{-\frac{t}{RC}}.$

(b) $q = Qe^{-\frac{t}{RC}}$, where Q is the charge at time $t = 0$.

(c) $q = CE + ce^{-\frac{t}{RC}}.$

(d) $q = ce^{\frac{t}{RC}} + \frac{CE}{1 + R^2C^2\omega^2}(\sin \omega t - RC\omega \cos \omega t).$

37. $s = \frac{\sqrt{2\kappa v_0{}^2t + 1} - 1}{\kappa v_0}.$

CHAPTER VI.

Art. 50.

3. $x = c_1e^{\frac{3}{2}t} + c_2e^{-4t}.$ **4.** $x = c_1e^{\frac{2}{3}x} + c_2e^{-\frac{8}{3}x}.$

Art. 51.

1. $y = e^{2x}(c_1 + c_2x) + c_3e^{-x}.$ **2.** $y = e^{-x}(c_1 + c_2x + c_3x^2) + c_4e^{4x}.$

Art. 52.

3. $y = e^x(c_1 + c_2x)\sin x + e^x(c_3 + c_4x)\cos x.$

Art. 58.

2. $y = c_1e^x + c_2e^{-x} - 2 - 5x.$ **3.** $y = e^{-x}(c_1 + c_2x) + \frac{2}{9}e^{2x}.$

4. $y = e^{2x}(c_1 + c_2x) + c_3e^{-3x} + e^{2x}\int\int e^{-5x}\int e^{3x}X(dx)^3.$

Art. 60.

3. $y = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + e^x(c_3 + \frac{2}{3}x) + \frac{1}{7}e^{2x} - 1.$

4. $y = e^x(ax + b) + \frac{4}{3}e^{\frac{5}{2}x}.$

Art. 61.

3. $y = c_1e^{-2x} + e^x(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x) + \frac{1}{8}(x^4 - x + 1).$

Art. 62.

3. $y = c_1e^{2x} + c_2e^{-2x} - \frac{8}{17}\sin\frac{1}{2}x.$

4. $y = ce^{-x} + e^{\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$

$$+\frac{\sin 3x + 27\cos 3x}{730} - \frac{1}{2} + \frac{\sin x - \cos x}{4}.$$

Art. 63.

2. $y = c_1e^{-x} + c_2e^{-2x} + \frac{e^{2x}}{170}(11\sin x - 7\cos x).$

3. $y = c\cos(\sqrt{2}\,x + a) + \frac{e^{3x}}{121}\left(11x^2 - 12x + \frac{50}{11}\right) + \frac{e^x}{17}(4\sin 2x - \cos 2x)$

Art. 64.

2. $y = c_1\cos(2x + a) + \frac{1}{3}x\sin x - \frac{2}{9}\cos x.$

3. $y = c_1e^x + c_2e^{-x} + x\sin x + \frac{1}{2}\cos x(1 - x^2).$

Page 80.

1. $y = (c_1e^x + c_2e^{-x})\cos x + (c_3e^x + c_4e^{-x})\sin x.$
2. $y = c_1e^{-x}\cos(x + a) + c_2e^{3x}\cos(2x + \beta) + c_3e^{-4x}.$
3. $y = c_1 + e^{-x}(c_2 + c_3x) + \frac{x^3}{3} - \frac{3x^2}{2} + 4x + \frac{e^{2x}}{18}.$
4. $y = c_1\cos 2x + c_2\sin 2x + \frac{1}{5}(e^x - \sin 3x) + \frac{1}{8}(2x^2 - 1).$
5. $y = c_1e^{2x} + c_2e^{3x} + \frac{1}{36}(6x + 5) + \frac{e^{mx}}{m^2 - 5m + 6}.$
6. $y = c_1e^{ax} + c_2e^{-ax} + \frac{e^{nx}}{n^2 - a^2} + \frac{xe^{ax}}{2a}.$
7. $y = c_1e^{-2x} + c_2e^{4x} + c_3e^x + \frac{1}{8}(x + \frac{3}{4}).$

8. $y = c_1 + c_2x + e^{-\frac{x}{2}}\left(c_3 \sin\frac{\sqrt{3}}{2}x + c_4 \cos\frac{\sqrt{3}}{2}x\right) + 3\,bx^2 - \frac{1}{3}\,ax^3$
$+ \frac{1}{12}(a - 3\,b)x^4 + \frac{1}{20}\,bx^5.$

9. $y = c_1e^x + c_2e^{3x} + c_3e^{-4x} + \frac{1}{12}\,(x + \frac{13}{12}).$

10. $y = c \sin\,(nx + \alpha)(ax + b) + \dfrac{\cos mx}{(n^2 - m^2)^2}.$

11. $y = (a + bx)\sin x + (c + dx)\cos x + \dfrac{x^3 \sin x}{12} + \dfrac{9\,x^2 - x^4}{48} \cos x.$

12. $y = c_1 \cos\,(ax + \alpha) + \dfrac{x \sin ax}{a} + \dfrac{\cos ax \log \cos ax}{a^2}.$

13. $y = (c + c_2x)e^x + \dfrac{e^{3x}}{8}\,(2\,x^2 - 4\,x + 3).$ $\quad - \dfrac{96(2\,x + 3)}{(n^2 + 1)^3} + \dfrac{384}{(n^2 + 1)^4}\Bigg].$

14. $y = c \cos\,(nx + \alpha) + \dfrac{e^x}{n^2 + 1}\Bigg[x^4 - \dfrac{4\,x^2(2\,x + 3)}{n^2 + 1} + \dfrac{24(2\,x^2 + 4\,x + 1)}{(n^2 + 1)^2}$

15. $y = c_1e^{ax} + c_2e^{-ax} + c_3 \sin\,(ax + \alpha) - a^{-4}x^4 - 24\,a^{-8}.$

16. $y = c_1 + c_2x + e^x(c_3 + c_4x) + x^2 + \frac{1}{6}\,x^3.$

17. $y = c_1e^x + c_2e^{-x} + c_3 \sin\,(x + \alpha) - \frac{1}{5}\,e^x \cos x.$

18. $y = e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(2 \cos 2\,x + 3 \sin 2\,x).$

19. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} - \dfrac{e^{2x}}{12}\left(x + \dfrac{17}{12}\right).$

20. $y = ce^x \sin\,(\sqrt{3}\,x + \alpha) + \frac{1}{2}\,e^x \cos x.$

21. $y = e^{-x}(c_1 + c_2x + c_3x^2) + \frac{1}{6}\,x^3e^{-x}.$

22. $y = e^{2x}(c_1 + c_2x) + e^{-x}(c_3 + c_4x) + \frac{1}{4}\,e^x(x^2 + 2\,x + \frac{7}{2}).$

23. $y = e^x(c_1 + c_2x + c_3x^2 + \frac{1}{6}\,x^3 + \frac{1}{24}\,x^4).$

24. $y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x \sin x + \cos x) + \frac{1}{12}\,xe^x(2\,x^2 - 3\,x + 9).$

25. $y = c_1e^{3x} + c_2e^x - \frac{1}{30}(2 \sin 3\,x + \cos 3\,x) - \dfrac{e^x}{8}\,(\sin 2\,x + \cos 2\,x).$

26. $y = e^x(c_1 + c_2 \cos x + c_3 \sin x) + xe^x + \frac{1}{10}(\cos x + 3 \sin x).$

27. $y = c_1e^{5x} + c_2e^{4x} + x + \frac{9}{20}.$

28. $y = e^{2x}(c_1 + c_2x) + c_3e^{-x} + \frac{1}{4}\,e^{3x}.$

29. $y = c_1e^{-x} + e^{\frac{x}{2}}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right).$
$+ \dfrac{e^{2x}}{130}\,(3 \sin x - 11 \cos x) - \frac{1}{6}\,xe^{\frac{x}{2}}\left(\sin\dfrac{x\sqrt{3}}{2} + \sqrt{3} \cos\dfrac{x\sqrt{3}}{2}\right).$

CHAPTER VII.

Art. 65.

2. $y = cx^{\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\log x + \alpha\right) + x^2.$

Art. 66.

3. $y = (c_1 + c_2\log x)\sin\log x + (c_3 + c_4\log x)\cos\log x.$

Art. 69.

2. $y = x^{-2}(c_1 + c_2\log x) + \frac{x^4}{36}.$ 3. $y = c_1x^{-5} + c_2x^4 - \left(\frac{x^2}{14} + \frac{x}{9} + \frac{1}{20}\right).$

Art. 71.

1. $y = (5 + 2x)^2[c_1(5 + 2x)^{\sqrt{2}} + c_2(5 + 2x)^{-\sqrt{2}}].$
2. $y = (2x - 1)[c_1 + c_2(2x - 1)^{\frac{\sqrt{3}}{2}} + c_3(2x - 1)^{\frac{-\sqrt{3}}{2}}].$

Page 91.

1. $y = c_1x^2 + x^{\frac{5}{2}}\left(c_2x^{\frac{\sqrt{21}}{2}} + c_3x^{\frac{-\sqrt{21}}{2}}\right) - \frac{x^3}{5}.$ 2. $y = c_1x^{-1} + c_2x^{-2} + x^{-2}e^x.$
3. $y = x^2[c_1 + c_2\log x + c_3(\log x)^2].$
4. $y = c_1(x + a)^2 + c_2(x + a)^3 + \frac{3x + 2a}{6}.$
5. $y = x^{-2}(c_1 + c_2\log x) + c_3x.$
6. $y = x(c_1\cos\log x + c_2\sin\log x + 5) + x^{-1}(c_3 + 2\log x).$
7. $y = [c_1 + c_2\log(x + 1)]\sqrt{x + 1} + \frac{c_3 + c_4\log(x + 1)}{\sqrt{x + 1}} + \frac{x^2 + 52x + 51}{225}.$
8. $y = c_1x + c_2x^{-1} + \frac{x^m}{m^2 - 1}.$
9. $y = x^2(c_1 + c_2\log x) + \frac{x^m}{(m - 2)^2}.$
10. $y =$ C.F. of Ex. 3, Art. 66, $+ (\log x)^2 + 2\log x - 3.$
11. $y = x(c_1 + c_2\log x) + c_3x^{-1} + \frac{1}{4}x^{-1}\log x.$
12. $y = \frac{1}{x}\left(\log\frac{x}{x - 1} + c_1\log x + c_2\right).$

13. $y = x^m(c_1 \sin \log x^n + c_2 \cos \log x^n + \log x)$.

14. $$y = x^2(c_1x^{\sqrt{3}} + c_2x^{-\sqrt{3}}) + \frac{1}{6\,x} + \frac{\log x}{61\,x}(5 \sin \log x + 6 \cos \log x) + \frac{2}{3721\,x}(27 \sin \log x + 191 \cos \log x).$$

CHAPTER VIII.

Art. 75.

3. $y = xe^{-2x}\left[\int c_1x^{-2}e^{2x}dx + c_3\right]$.

4. $e^{2e^x}y = \frac{1}{3}\int e^{2e^x}x^3dx + c_1\int e^{2e^x}dx + c_2$.

5. $e^{\frac{4}{3}x^{\frac{3}{2}}}\frac{y}{x} = \frac{1}{5}e^{\frac{4}{3}x^{\frac{3}{2}}} + c_1\int e^{\frac{4}{3}x^{\frac{3}{2}}}\frac{dx}{x^2} + c_2$.

6. $dy = \sqrt{x^2y^2 + c^2}dx$. **7.** $xy^2 + c_1x^5 = c_2$.

Art. 76.

2. $y = c_1 + c_2x + c_3x^2 + \cdots + c_nx^{n-1} + \frac{|\underline{m}\,x^{m+n}}{|\underline{m+n}}$.

3. $y = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{1}{2}x^2 \log x$.

4. $y = c_1 + c_2x + (6 - x^2) \sin x - 4\,x \cos x$.

Art. 77.

2. $3\,x = 2\,a^{\frac{1}{4}}(y^{\frac{1}{2}} - 2\,c_1)(y^{\frac{1}{2}} + c_1)^{\frac{1}{2}} + c_2$.

3. $\sqrt{c_1y^2 + y} - \frac{1}{\sqrt{c_1}}\log(\sqrt{c_1y} + \sqrt{1 + c_1y}) = ac_1\sqrt{2}\,x + c_2$.

4. $ax = \log(y + \sqrt{y^2 + c_1}) + c_2$, or $y = c_1'e^{ax} + c_2'e^{-ax}$.

Art. 78.

2. $2\,(y - b) = e^{x \quad a} + e^{-(x-a)}$. **3.** $y = c_1x + (c_1^2 + 1) \log (x - c_1) + c_2$

4. $15\,c_1^2y = 4\,(c_1x + a^2)^{\frac{5}{2}} + c_2x + c_3$.

Art. 79.

1. $e^{-ay} = c_1x + c_2$.

2. $y^2 = x^2 + c_1x + c_2$.

3. $\log y = c_1e^x + c_2e^{-x}$.

4. $\sin\,(c_1 - 2\sqrt{2}\,y) = c_2e^{-2x}$.

Art. 80.

1. $y = c_1 \sin ax + c_2 \cos ax + c_3x + c_4$.

2. $y = c_1e^{mx} + c_2e^{-mx} + c_3 + c_4x + c_5x^2 + \frac{e^{ax}}{a^3(a^2 - m^2)}$.

3. $y = c_1 + c_2x + x^{\frac{5}{2}}\left(c_3x^{\frac{\sqrt{1-4a^2}}{2}} + c_4x^{-\frac{\sqrt{1-4a^2}}{2}}\right)$ when $a < \frac{1}{2}$,

$y = c_1 + c_2x + c_3x^{\frac{5}{2}} \cos\left(\frac{\sqrt{4\,a^2 - 1}}{2} \log \frac{x}{c_4}\right)$ when $a > \frac{1}{2}$.

Art. 81.

2. $\frac{2\,c_1y}{a} = c_1{}^2e^{\frac{x}{a}} + e^{-\frac{x}{a}} + c_2.$

3. $y = c_1 \log x + c_2.$

4. $15\,y = 8(x + c_1)^{\frac{5}{2}} + c_2x + c_3.$

Art. 82.

4. $$y = A\left(1 - \frac{2}{\underline{|2}}x^2 + \frac{2(2\cdot1\cdot2+4)}{\underline{|4}}x^4 - \frac{2(2\cdot1\cdot2+4)(2\cdot3\cdot4+6)}{\underline{|6}}x^6 + \cdots\right)$$
$$+ B\left(x - \frac{3}{\underline{|3}}x^3 + \frac{3(2\cdot2\cdot3+5)}{\underline{|5}}x^5 - \frac{3(2\cdot2\cdot3+5)(2\cdot4\cdot5+7)}{\underline{|7}}x^7 + \cdots\right).$$

5. $$y = Ax\left(1 + \frac{x^2}{2\cdot5} + \frac{x^4}{2\cdot4\cdot5\cdot9} + \frac{x^6}{2\cdot4\cdot6\cdot5\cdot9\cdot13} + \cdots\right)$$
$$+ Bx^{\frac{1}{2}}\left(1 + \frac{x^2}{2\cdot3} + \frac{x^4}{2\cdot4\cdot3\cdot7} + \frac{x^6}{2\cdot4\cdot6\cdot3\cdot7\cdot11} + \cdots\right)$$
$$+ \frac{x^2}{1\cdot3} + \frac{x^4}{1\cdot3\cdot3\cdot7} + \frac{x^6}{1\cdot3\cdot5\cdot3\cdot7\cdot11} + \cdots.$$

6. $$y = A\left(1 - n\cdot n+1\cdot\frac{x^2}{\underline{|2}} + n\cdot n-2\cdot n+1\cdot n+3\cdot\frac{x^4}{\underline{|4}} - \cdots\right)$$
$$+ B\left(x - n-1\cdot n+2\frac{x^3}{\underline{|3}} + n-1\cdot n-3\cdot n+2\cdot n+4\cdot\frac{x^5}{\underline{|5}} - \cdots\right),$$
$$y = Ax^n\left(1 - \frac{n\cdot n-1}{2\cdot 2\,n-1}x^{-2} + \frac{n\cdot n-1\cdot n-2\cdot n-3}{2\cdot4\cdot2\,n-1\cdot2\,n-3}x^{-4} - \cdots\right)$$
$$+ Bx^{-n-1}\left(1 + \frac{n+1\cdot n+2}{2\cdot2\,n+3}x^{-2} + \frac{n+1\cdot n+2\cdot n+3\cdot n+4}{2\cdot4\cdot2\,n+3\cdot2\,n+5}x^{-4} + \cdots\right).$$

Page 107.

1. $y^2\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 + xy = c^2.$

2. $2\,ay + x^2 = c_1\sqrt{a^2 - x^2} + c_2.$

3. $(1 + x + x^2)y = c_1x^2 + c_2x + c_3.$

4. $x^3\frac{d^2y}{d^2x} + x^2\frac{dy}{dx} + x^3y = x^2 + c.$

5. $y = c_1 + c_2x + c_3e^{ax} + c_4e^{ax}.$

6. $c_1y = c_2e^{c_1x} - n\sqrt{1 + a^2c_1{}^2}.$

7. $xy\sqrt{x^2 - 1} = \sec^{-1}x + c_1\sqrt{x^2 - 1} + c_2\log(x + \sqrt{x^2 - 1}) + c_3.$

8. $y = c_2 - \sin^{-1}c_1e^{-x}.$

9. $y = c_1\sin^{-1}x + (\sin^{-1}x)^2 + c_2$

10. $y = c_1 \sin^2 x + c_2 \cos x - c_2 \sin^2 x \log \tan \frac{x}{2}.$

11. $y = c_2 e^x (x - 1) + c_1 x + c_1 \int x e^x \int x^{-1} e^{-x} (dx)^2 + c_3.$

12. $y = ax \log x + c_1 x + c_2.$ **14.** $y = c_1 x^2 + c_2 x + c_3 + \frac{x^3}{12} + \frac{\sin 2x}{16}.$

13. $\log y - 1 = \frac{1}{c_1 x + c_2}.$ **15.** $y = c_1 e^{-x} + c_2 + \frac{1}{2} e^x.$

16. $y = e^{-\sin x} \int e^{\sin x} (c_1 x + c_2) dx + c_3 e^{-\sin x} - \frac{\sin x - 1}{2}.$

17. $y = c_1 (1 - x \cot x) + c_2 \cot x.$ **18.** $a \log (y + b) = x + c.$

19. $(c_1 x + c_2)^2 + a = c_1 y^2.$

20. $\frac{d^2y}{dx^2} = \int f(x) dx + c_1, \qquad x \frac{d^2y}{dx^2} - \frac{dy}{dx} = \int x f(x) dx + c_2,$

$$x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = \int x^2 f(x) dx + c_3.$$

21. $y - b = \frac{1}{\kappa^2} \log \sec a\kappa (x - c).$

CHAPTER IX.

Art. 87.

2. $y = Ax + Bx \int x^{-2} e^{\frac{1}{3}x^3} dx + 1.$

3. $y = Ae^x + Be^{3x}(4x^3 - 42x^2 + 150x - 183).$ **4.** $y = \frac{c}{x} + c_1 \left(x + \frac{1}{x} \right).$

Art. 88.

3. $y = c_1 e^{2x} + c_2 e^{2x} \int e^{\frac{2}{3x} - 2x} dx.$ **4.** $y = c_1 e^{\frac{2}{3x}} + c_2 e^{\frac{2}{3x}} \int x^{-2} e^{2x - \frac{2}{3x}} dx.$

Art. 91.

2. $y = ce^{-\frac{x^4}{8}} x^{\frac{1}{2}} \sin \left(\frac{\sqrt{3}}{2} \log x + a \right).$

3. $y = (c_1 \sin \sqrt{6}\, x + c_2 \cos \sqrt{6}\, x) \sec x.$ **4.** $y = e^x (c_1 x^2 + c_2 x)$

Art. 92.

2. $y = c \sin \left(2 \log \tan \frac{x}{2} + a \right).$ **3.** $y = c_1 \sin (x^2 + a) + \frac{x^2}{4}.$

4. $y = c_1 \cos \frac{a}{2x^2} + c_2 \sin \frac{a}{2x^2} + \frac{1}{a^2 x^2}.$

Page 120.

1. $y = \frac{1}{x}(c_1 e^{nx} + c_2 e^{-nx})$.

2. $xy = c \sin (nx + \alpha)$.

3. $y = cx \sin (nx + \alpha)$.

4. $y\sqrt{1 + x^2} = c_1 \log (x + \sqrt{1 + x^2}) + c_2$.

5. $y = c_1 e^x + c_2 e^{3x}(4x^3 - 42x^2 + 150x - 183)$.

6. $y = ce^{\frac{1}{2}bx^2} \sin (x\sqrt{b} + \alpha)$.

7. $y = e^{-x^2}(c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}})$.

8. $y = e^x(c_1 \log x + c_2)$.

9. $y = c_1 x + c_2(x \sin^{-1} x + \sqrt{1 - x^2}) - \frac{1}{9}x(1 - x^2)^{\frac{3}{2}}$.

10. $y = c_1 x + c_2 \cos x$.

11. $y = c_1 x^2 + c_2 x + c_3 \left(x^2 \int x^{-3} e^{-x} dx - x \int x^{-2} e^{-x} dx \right)$.

12. $y = c_1 e^{a \sin^{-1} x} + c_2 e^{-a \sin^{-1} x}$.

13. $y = c_1 x + c_2 x \int x^{-2} e^{-\frac{x^2}{2}} dx + x \int x^{-2} e^{-\frac{x^2}{2}} \int x e^{\frac{x^2}{2}} f(x)(dx)^2$.

14. $2y = x(c_1 e^{2x} + c_2 - x)$.

15. $y = c_1 \sin \left(\sqrt{\frac{a^2 - x^2}{a}} + \alpha \right)$.

16. $y = c_1 \sin (n\sqrt{x^2 - 1} + \alpha)$.

17. $y^2 = cx^2 + c_1 x$.

18. $y^2 + (x - c)^2 = k^2$.

19. $y = c \cos \left(\frac{n + ax}{x} \right)$.

CHAPTER X.

Page 124.

1. The circle of radius $\frac{1}{\kappa}$.

2. A catenary, $y = \frac{c}{2}\left(e^{\frac{x + c'}{c}} + e^{-\frac{x + c'}{c}} \right)$.

3. $y^2 + (x - a)^2 = c^2$, circles whose centres are on the x-axis.

4. $(x - a)^2 = 4c(y - c)$, a system of parabolas whose axes are parallel to the axis of y.

5. $x + c_1 = c \operatorname{vers}^{-1}\frac{y}{c} - \sqrt{2cy - y^2}$, the cycloid obtained by rolling any circle along the x-axis from any point.

6. The ellipses $a^2y^2 + \kappa^2(x - c)^2 = a^4$, if the cube of the normal is $-\kappa^2$ times the radius of curvature.

The hyperbolas $a^2y^2 - a^4(x - c)^2 = \kappa^2$, if the cube of the normal is $+\kappa^2$ times the radius of curvature.

A set of parabolas if no constant is introduced at the first integration.

7. The elastic curve represented by the equation

$$\{4\kappa^2-(x^2-a^2)^2\}^{\frac{1}{2}}dy=\pm(x^2-a^2)dx.$$

8. $s=c_1e^{\kappa t}+c_2e^{-\kappa t}$, when accel $=\kappa^2$ distance from the fixed point.
$s=c_1\sin(\kappa t+c_2)$, when accel $=-\kappa^2$ distance from the fixed point.

9. $s=\frac{1}{2}at^2+v_0t+s_0$.

10. The relation between time of motion and the distance passed over is $t=c_1\pm\frac{\sqrt{c}}{\sqrt{2}\,\kappa}\{\sqrt{s^2-cs}+c\log(\sqrt{s}+\sqrt{s-c})\}$, according as the acceleration is $\pm\frac{\kappa^2}{s^2}$.

11. $s=\frac{g}{n^2}\log\cosh nt$, if the resistance of the air is $\frac{n^2}{g}$ times the square of the velocity.

12. $s=\frac{\sqrt{2\kappa v_0^2t+1}-1}{\kappa v_0}$, if the acceleration is $-\kappa$ times the cube of the velocity.

13. $T=2\pi\sqrt{\frac{A}{MH}}$. (Emtage, *Mathematical Theory of Electricity and Magnetism*, p. 85.)

14. $s=l+(s_0-l)\cos\kappa t$, $v=-\kappa(s_0-l)\sin\kappa t$, where $\kappa=\sqrt{\frac{g}{e}}$.

HINT: Put $s-l$ equal to a new variable.

16. $\frac{a^2}{\sqrt{\mu}}$, if acceleration is $-\frac{\mu}{x^3}$.

17. $$i=\frac{C}{\sqrt{R^2C^2-4LC}}\left\{e^{-\frac{t}{T_1}}\int e^{\frac{t}{T_1}}f'(t)dt-e^{-\frac{t}{T_2}}\int e^{\frac{t}{T_2}}f'(t)dt\right\}$$
$$+c_1e^{-\frac{t}{T_1}}+c_2e^{-\frac{t}{T_2}},$$
where $T_1=\frac{2LC}{RC-\sqrt{R^2C^2-4LC}}$, and $T_2=\frac{2LC}{RC+\sqrt{R^2C^2-4LC}}$.

18. Same as in 17 with $f(t)$ substituted for $f'(t)$ and q for i.

19. $i=e^{-\frac{Rt}{2L}}(c_1+c_2t)$.

20. $i=I\sin\frac{t}{\sqrt{LC}}$.

21. $\theta-\alpha=c_1e^{-t\kappa}\sin(\sqrt{\omega^2-\kappa^2}\,t+c_2)$ for $\omega^2>\kappa^2$;
$\theta-\alpha=c_1e^{-(\kappa-\sqrt{\kappa^2-\omega^2})t}+c_2e^{-(\kappa+\sqrt{\kappa^2-\omega^2})t}$ for $\omega^2<\kappa^2$.

22. $6EIy=P(3l^2x-x^3)$.

23. $24EIy=w(4l^3x-x^4)$.

24. The general solution is

$$y = A \cosh \sqrt{\frac{Q}{EI}}\, x + B \sinh \sqrt{\frac{Q}{EI}}\, x + \frac{wx^2}{2\,Q} + \frac{wEI}{Q^2}.$$

On applying the conditions of Ex. 22 to determine the constants,

$$A = -\frac{WEI}{Q^2}, \quad B = 0;$$

and therefore, $$y = \frac{WEI}{Q^2}\left(1 - \cosh \sqrt{\frac{Q}{EI}}\, x\right) + \frac{wx^2}{2\,Q}.$$

CHAPTER XI.

Art. 98.

2. $x = e^{6t}(A \cos t + B \sin t), \quad y = e^{6t}[(A - B) \cos t + (A + B) \sin t].$

3. $x = c_1 e^{-5t} + c_2 e^t + \frac{2}{7} e^{2t} - \frac{2}{5} t - \frac{13}{25}, \quad y = -c_1 e^{-5t} + c_2 e^t + \frac{4}{7} e^{2t} - \frac{3}{5} t - \frac{12}{25}.$

4. $x = c_1 e^{-t} + c_2 e^{-6t} + \frac{19}{3} t - \frac{56}{9} - \frac{29}{7} e^t, \quad y = -c_1 e^{-t} + 4 c_2 e^{-6t} - \frac{17}{3} t + \frac{55}{9} + \frac{24}{7} e^t.$

5. $x = (c_1 + c_2 t) e^t + (c_3 + c_4 t) e^{-t}; \quad 2y = (c_2 - c_1 - c_2 t) e^t - (c_3 + c_4 + c_4 t) e^{-t}.$

Art. 99.

3. $x^2 = z^2 + c_1, \; x^3 = y^3 + c_2.$ **4.** $z = \frac{nxy}{y - x} \log \frac{y}{x} + c_1, \; y - x = cxy.$

5. $x^3 - y^3 = c_1^3, \; x^3 + \frac{3}{z} = c_2.$

6. $ax^2 + by^2 + cz^2 = c_1, \; a^2x^2 + b^2y^2 + c^2z^2 = c_2.$

Art. 103.

3. $(y + z)e^x = c.$ **4.** $x - cy - y \log z = 0.$

5. $e^{x^2}(y + z^2 + x) = c.$ **6.** $y(x + z) = c(y + z).$

Page 143.

1. $x = c_1 e^{-\frac{6}{5}t} + \frac{1}{2} e^{2t} - \frac{3}{11} e^t, \; x + \frac{y}{8} = c_2 e^{-t} + \frac{1}{2} e^{2t} - \frac{3}{16} e^t.$

2. $x = (c_1 \sin t + c_2 \cos t) e^{-4t} + \frac{31\, e^t}{26} - \frac{93}{17},$

$y = [(c_2 - c_1) \sin t - (c_2 + c_1) \cos t] e^{-4t} - \frac{2\, e^t}{13} + \frac{6}{17}.$

3. $y = (c_1 + c_2 x) e^x + 3 c_3 e^{-\frac{3}{2}x} - \frac{1}{2} x, \; z = 2(3 c_2 - c_1 - c_2 x) e^x - c_3 e^{-\frac{3}{2}x} - \frac{1}{3}.$

4. $x = c_1 e^{-2t} + c_2 e^{-7t} - \frac{31}{196} + \frac{5}{14} t - \frac{1}{8} e^t,$

$y = -\frac{2}{3} c_1 e^{-2t} + c_2 e^{-7t} + \frac{9}{98} - \frac{1}{7} t + \frac{5}{24} e^t.$

5. $x = \frac{t}{3} + \frac{a}{t^2}, \ x + y = be^t.$

6. $x^2+y^2+z^2=c_1, \ x^2+y^2-z^2=c_2.$

7. $\frac{y+z}{x} + \frac{z+x}{y} = c.$

8. $xy + yz + zx = c(x + y + z).$

9. $\log xyz + x + y + z = c.$

10. $(y + z)(u + c) + z(x - u) = 0.$

11. $x^2 + xy^2 - u + x^2 z = c.$

12. $z^2 + (x - a)^2 + (y - b)^2 = h^2.$

13.
$$\begin{cases} x = c_1 \cos \sqrt{3}\, t + c_2 \sin \sqrt{3}\, t + c_3 \cos \sqrt{2}\, t + c_4 \sin \sqrt{2}\, t \\ \quad - \frac{49}{1452} e^{3t} + \frac{5}{66} te^{3t} - \frac{1}{12} - \frac{1}{4} \cos 2\, t, \\ y = -3\, c_1 \cos \sqrt{3}\, t - c_2 \sin \sqrt{3}\, t - 2\, c_3 \cos \sqrt{2}\, t - 2\, c_4 \sin \sqrt{2}\, t \\ \quad + \frac{1}{66} te^{3t} + \frac{1}{3} - \frac{23}{1452} e^{3t}. \end{cases}$$

14.
$$x = e^{\frac{mt}{\sqrt{2}}}\left(c_1 \cos \frac{mt}{\sqrt{2}} + c_2 \sin \frac{mt}{\sqrt{2}}\right) + e^{-\frac{mt}{\sqrt{2}}}\left(c_3 \cos \frac{mt}{\sqrt{2}} + c_4 \sin \frac{mt}{\sqrt{2}}\right).$$
$$y = e^{\frac{mt}{\sqrt{2}}}\left(c_1 \sin \frac{mt}{\sqrt{2}} - c_2 \cos \frac{mt}{\sqrt{2}}\right) + e^{-\frac{mt}{\sqrt{2}}}\left(c_4 \cos \frac{mt}{\sqrt{2}} - c_3 \sin \frac{mt}{\sqrt{2}}\right).$$

15.
$$\begin{cases} x = a_1 \sin \kappa t + a_2 \cos \kappa t + a_3, \\ y = b_1 \sin \kappa t + b_2 \cos \kappa t + b_3, \\ z = c_1 \sin \kappa t + c_2 \cos \kappa t + c_3, \end{cases}$$

where $\kappa^2 = l^2 + m^2 + n^2$; and the arbitrary constants are connected by the following relations:

$$\frac{mc_1 - nb_1}{a_2} = \frac{na_1 - lc_1}{b_2} = \frac{lb_1 - ma_1}{c_2} = k,$$
$$la_1 + mb_1 + nc_1 = 0, \ \frac{a_3}{l} = \frac{b_3}{m} = \frac{c_3}{n}.$$

16. See Forsyth, *Diff. Eq.*, Ex. 3, Art. 174; Johnson, *Diff. Eq.*, Art. 242.

17. $x + m_1 y = c_1 e^{(a+m_1 a')^{\frac{1}{2}} t} + c_2 e^{-(a+m_1 a')^{\frac{1}{2}} t},$
$x + m_2 y = c_3 e^{(a+m_2 a')^{\frac{1}{2}} t} + c_4 e^{-(a+m_2 a')^{\frac{1}{2}} t},$
where m_1 and m_2 are the roots of $a'm^2 + (a - b)m - b = 0.$
Ex. 16, p. 269, Johnson, *Diff. Eq.*; Ex. 4, p. 270, Forsyth, *Diff. Eq.*

18. When the horizontal and vertical lines through the starting point in the plane of motion are taken for the x and y axes, the equation of the path is

$$x = v_0 t \cos \phi, \ y = v_0 t \sin \phi - \tfrac{1}{2} gt^2;$$

and the elimination of t gives the parabola $y = x \tan \phi - \frac{1}{2} g \frac{x^2}{v_0^2 \cos^2 \phi}.$

19. Axes being chosen as in Ex. 18,

$$x = \frac{v_0}{c} \cos \phi (1 - e^{-ct}), \ y = -\frac{g}{c} t + \frac{cv_0 \sin \phi + g}{c^2}(1 - e^{-ct}).$$

20. For upper sign: the hyperbola $(a_1y-b_1x)(b_2x-a_2y)=(a_1b_2-a_2b_1)^2$.
For lower sign: the ellipse $(a_1y-b_1x)^2+(a_2y-b_2x)^2=(a_1b_2-a_2b_1)^2$.

CHAPTER XII.

Art. 108.

2. $z=px+qy+pq$. **3.** $p=q$. **4.** $q=2yp^2$. **5.** $z=pq$.

6. $xz\dfrac{\partial^2 z}{\partial x^2}+x\left(\dfrac{\partial z}{\partial x}\right)^2-z\dfrac{\partial z}{\partial x}=0$, or $yz\dfrac{\partial^2 z}{\partial y^2}+y\left(\dfrac{\partial z}{\partial y}\right)^2-z\dfrac{\partial z}{\partial y}=0$.

Art. 109.

2. $yp-xq=0$. **3.** $(l+np)y+z(lq-mp)=(m+nq)x$.

4. $x^2\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=2y^2$. **5.** $\dfrac{\partial^2 z}{\partial y^2}=a^2\dfrac{\partial^2 z}{\partial x^2}$.

Art. 116.

2. $z=e^{\frac{y}{a}}\phi(x-y)$. **3.** $lx+my+nz=\phi(x^2+y^2+z^2)$.

4. $\dfrac{1}{x}-\dfrac{1}{y}=f\left(\dfrac{1}{x}-\dfrac{1}{z}\right)$. **5.** $f(x^2-z^2,\ x^3-y^3)=0$.

Art. 117.

2. $xyz-3u=\phi\left(\dfrac{y}{x},\dfrac{x}{z}\right)$.

Art. 119.

3. $z=a\left(x+\dfrac{3}{2+\sqrt{10}}y\right)+c$. **4.** $z=mx+ye^{-\frac{m}{a}}+c$

5. $z=ax+\dfrac{ky}{a}+b$.

Art. 120.

2. $z=ax+by-2\sqrt{ab}$.

Art. 121.

2. $z^{\frac{2a^2}{\sqrt{4a^2+1}-1}}=bxy^a$. **3.** $(z+a^2)^3=(x+ay+c)^2$.

4. $4c(z-a)=(x+cy+b)^2+4$.

5. $z^2\pm\left[z\sqrt{z^2-4a^2}-4a^2\log\left(z+\sqrt{z^2-4a^2}\right)\right]=4(x+ay+b)$.

Art. 122.

2. $z^{\frac{3}{2}}=(x+a)^{\frac{3}{2}}+(y+a)^{\frac{3}{2}}+b.$ **4.** $z=\frac{1}{6}(2x-a)^3+a^2y+b.$

3. $z=ax+a^2y^2+b.$ **5.** $z=\frac{2}{3}(x+a)^{\frac{3}{2}}+\frac{2}{3}(y-a)^{\frac{3}{2}}+b$

6. $z^2=x\sqrt{x^2+a^2}+y\sqrt{y^2-a^2}+a^2\log\dfrac{x+\sqrt{x^2+a^2}}{y+\sqrt{y^2-a^2}}+b.$

Art. 123.

2. $z=(x+a)(y+b)$, S.I. is $z=0$.

Another form of the C.I. is $2\sqrt{z}=\dfrac{x}{a}+ay+b.$

3. C.I. is $z^2=a^2y^2+(ax+b)^2$.

Art. 125.

3. $az=\dfrac{x^3y}{6}+x\phi(y)+\psi(y).$ **4.** $z=F(y)x^n+f(y).$

Art. 126.

2. $x=f(z)+\phi(y).$ **3.** $z=xf\left(\dfrac{y}{x}\right)+F\left(\dfrac{y}{x}\right).$

Art. 128.

2. $z=\phi(y-2x)+\psi(y-x).$ **3.** $z=\phi(y+3x)+\psi(y-2x).$

Art. 129.

1. $z=x^2\phi_1(y+x)+x\phi_2(y+x)+\phi_3(y+x).$

Art. 130.

3. $\dfrac{x^4}{12}.$ **4.** $\dfrac{x^3y}{6}+\frac{1}{24}x^4.$

Art. 131.

4. $z=\phi(x+y)+e^{3y}\psi(y-x).$ **5.** $z=e^x\phi(y)+e^{-x}\psi(x+y)$

6. $z=e^x\phi(y-x)+e^{-2x}\psi(y+x).$

Art. 132.

3. $-(ye^{x+2y}+\frac{1}{18}x^3+\frac{1}{6}x^2y+\frac{1}{9}x^2+\frac{1}{9}xy+\frac{2}{27}x)$, $\frac{1}{2}\sin(x+2y)-xe^y$,

$-\frac{1}{4}e^{x-y}+\frac{1}{2}x^2y+\frac{3}{4}x^2+\frac{1}{2}xy+\frac{3}{2}x+\frac{3}{4}y+\frac{21}{8}.$

Art. 133.

2. $\phi(x^2y) + x\psi(x^2y) + \frac{x^3y^4}{30}$.

3. $z = \phi(x^2 + y^2) + \psi(x^2 - y^2)$. (Put $\bar{x}$ for $\frac{1}{2}x^2$, $\bar{y}$ for $\frac{1}{2}y^2$.)

Page 187.

1. $z = x^n\phi\left(\frac{y}{x}\right)$.

2. $x^2 + y^2 + z^2 = z\phi\left(\frac{y}{z}\right)$.

3. $(a-1)z + \frac{xy}{t} = x^a\phi\left(\frac{y}{x}, \frac{t}{x}\right)$.

4. $\frac{y-b}{z-c} = \phi\left(\frac{x-a}{z-c}\right)$.

5. $(z-y)\sqrt{x+y+z} = \phi\left(\frac{z-y}{x-z}\right)$.

6. $a\log z = acx + (1-ac)y + b$.

7. $z = \frac{1}{x^3y^3}\phi\left(\frac{x}{y^2} + \frac{y}{x^2}\right)$.

8. $z + \sqrt{x^2+y^2+z^2} = x^{1-a}\phi\left(\frac{y}{x}\right)$.

9. $\phi\left(\frac{x-y}{y-z}, \frac{z-x}{y-z}\right) = 0$.

10. $x + y + z = \phi(xyz)$.

11. $\{\cos(x+y) + \sin(x+y)\}e^{y-x} = \phi\left\{z^{\sqrt{2}}\tan\left(\frac{3\pi}{8} - \frac{x+y}{2}\right)\right\}$.

12. $\sqrt{1+az^2} - \log\frac{1+\sqrt{1+az^2}}{z} = x + ay + b$.

13. $xz = ay + 2\sqrt{ax} + b$.

14. $(z - b - a\log x)^2 = 4a^2y$.

15. $z = ax + \frac{2a}{n \pm \sqrt{n^2-4}}y + c$.

16. $z = ax + by + c\sqrt{1+a^2+b^2}$. S.I. is $x^2 + y^2 + z^2 = c^2$

17. $z = ax + (1-\sqrt{a})^2y + b$.

18. $z = axe^y + \frac{1}{2}a^2e^{2y} + b$.

19. $az - 1 = ce^{x+ay}$.

20. (1) $2z = \left(\frac{x}{a} + ay\right)^2 + b$; (2) $z = xy + y\sqrt{x^2 - a_1{}^2} + b_1$;
(3) $z = xy + x\sqrt{y^2 + a_2{}^2} + b_2$.

21. $z = \frac{1}{2}a\log(x^2+y^2) + \sqrt{1-a^2}\tan^{-1}\frac{y}{x} + b$. (Change to polar co-ordinates.)

22. $\frac{2l}{2-l}z^{1-\frac{1}{2}l} = \frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{n+1}$.

23. $z = a\sqrt{x+y} + \sqrt{1-a^2}\sqrt{x-y} + b$. (Put $\sqrt{x+y} = u$, $\sqrt{x-y} = v$.)

24. $z = axy + a^2(x+y) + b$. (Put $xy = v$, $x + y = w$.)

25. $\sqrt{1+a}\,z = 2\sqrt{x+ay} + b.$

26. $z = x^3y^2 + \phi(y)\log x + \psi(y).$

27. $zy^2 = xy + F(x) + \psi(y).$

28. $y = x\phi(z) + \psi(z).$

29. $x = \phi(z) + \psi(x+y+z).$

30. $(n-1)zy + ax = e^{(n-1)x}\psi(y) + f(y).$

31. $z = \int e^{-\int f(x)dx}\left[e^{\int f(x)dx}F(y)dx + \phi(y)\right]dx + \psi(y).$

32. $z = \frac{1}{2}yx^2\log x + x^2\phi(y) + \psi(y).$

33. $z = \frac{1}{2}x^2y - xy + \phi(y) + e^{-x}\psi(y).$

34. $4z = x^2y^2 + \phi(y) + \psi(x).$

35. $z = -\frac{a+b}{24}x^4 + \frac{x^3y}{6} + F(y-bx) + f(y-ax).$

36. $y + x\phi(ax+by+cz) = \psi(ax+by+cz).$

37. $3z = ax(y^2-1)(y^2+2) + \phi(x)\sqrt{1-y^2} + \psi(y).$

38. $4z = 5x^3y^2 + \int\phi\left(\frac{y}{x}\right)dx + \psi(y).$

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